Efficient MHD equilibrium solver for Control Oriented Transport models

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Outline

1 Motivation: Physics-based-models for current profile control
2 Magnetic Diffusion Equation & Control Oriented Transport
3 Equilibrium constrain: the Grad-Shafranov equation
4 A $q$-solver algorithm
5 Convergence & Performance
6 Summary and perspective
Plasma control: Control categories and physical actuators in ITER

Control categories
- Plasma equilibrium
- Plasma current
- Vertical stability
- Burn state
- Divertor
- Current profile
- MHD instabilities
- Fast particles
- Error field
- Disruption mitigation

Actuators
- PF coils
- CS coils
- ECCD
- ECRH, ICRH
- NBI
- VS3 coils
- RMP coils
Elements of control process

Motivation
Physics-based-models for current plasma profile control
Magnetic Diffusion Equation (MDE)

\[
\frac{\partial \psi}{\partial t} = \frac{\eta(T_e)}{\mu_0 \rho_b^2 \hat{F}} \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left( \hat{\rho} \hat{F} \hat{G} \hat{H} \frac{\partial \psi}{\partial \hat{\rho}} \right) + R_0 \hat{H} \eta(T_e) \frac{\langle \mathbf{j}_{NI} \cdot \mathbf{B} \rangle}{B_{\phi,0}},
\]

\[
\left. \frac{\partial \psi}{\partial \hat{\rho}} \right|_{\hat{\rho}=0} = 0, \quad \left. \frac{\partial \psi}{\partial \hat{\rho}} \right|_{\hat{\rho}=1} = -\frac{\mu_0}{2\pi} \frac{R_0}{\hat{G}} \left. \hat{H} \right|_{\hat{\rho}=1} \left. I(t) \right|_{\hat{\rho}=1},
\]

where

\[
\hat{F}(\hat{\rho}) = \frac{R_0 B_{\phi,0}}{RB_{\phi}}, \quad \hat{G}(\hat{\rho}) = \left\langle \frac{R_0^2}{R^2} |\nabla \rho|^2 \right\rangle, \quad \hat{H}(\hat{\rho}) = \frac{\hat{F}}{\left\langle \frac{R_0^2}{R^2} \right\rangle},
\]

are “magnetic geometric” factors determined by plasma equilibrium.

In current implementations, these profiles are externally imposed, and typically left invariant.

\( q \) profile control and good discharge reproducibility have been achieved with this simplification [e.g. Schuster NF 57 116026 (2017), Felici NF 58 096006 (2018)].

However, a self-consistent description would allow more general and robust control design.
Coupling MDE with the Grad-Shafranov equation (GSE)

Simplified staggered scheme for Control Oriented Transport models

\[
\psi_i(\rho) - \frac{1}{2}(R, Z) \psi_i + \frac{1}{2}(R, Z) \psi_i + 1(\rho) = n_i(\rho), T_i(\rho), n_i + 1(\rho), T_i + 1(\rho) \]

\[
q_i(\rho), p_i(\rho), q_i, p_i, \{F, G, H\}, \{\hat{F}, \hat{G}, \hat{H}\} \]

\[
\{j_{aux}, j_{bs}\} \text{ (non-inductive currents)}
\]

Particle & power sources

Mass & energy transport

GSE
Plasma Equilibrium (prescribed boundary)

MDE
Magnetic Flux transport

\[
\psi_{i-1/2}(R, Z) \rightarrow \psi_i(\rho) \rightarrow \psi_{i+1/2}(R, Z)
\]

\[
\{F, G, H\} \rightarrow \{\hat{F}, \hat{G}, \hat{H}\}
\]

\[
\{j_{aux}, j_{bs}\}
\]

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GSE as a non-linear eigenvalue problem

\[ \Delta^* \Psi = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{1}{2} \frac{dF^2}{d\Psi}, \quad F = RB_\phi \]

Using the normalization, \( \psi = \Psi / \Psi_0 \), \( \Psi_0 \) poloidal flux function on axis, and defining \( f(\psi) = F(\Psi) \), \( p(\psi) = P(\Psi) \)

the (non-linear) eigenvalue nature of the equation is revealed
[LoDestro PoP 1 90 (1994)]

\[ \begin{cases} 
-\Delta^* \psi = \frac{1}{\Psi_0^2} \left( R^2 \frac{dp}{d\psi} + \frac{1}{2} \frac{df^2}{d\psi} \right) & \text{in } \Omega \\
\psi|_{\partial\Omega} = 0 & \text{0} \leq \psi \leq 1
\end{cases} \]

Methods to solve this problem are available e.g. [Pataki JCP 243 28 (2013)]

However, they require specification of \( p(\psi), f(\psi) \) instead of \( p(\rho), q(\rho) \) !!!
q-solver algorithm (eulerian description)

(0) A good seed: the linear eigenvalue solution

\[
\begin{align*}
- \Delta^* \psi &= \left( R^2 \mathcal{L}_p + \mathcal{L}_f \right) \psi \quad \text{in } \Omega \\
\psi \big|_{\partial \Omega} &= 0
\end{align*}
\]

($\mathcal{L}_p, \mathcal{L}_f$) are chosen to match prescribed ($I_p, \beta$) $\rightarrow \Psi^{k=0}(R, Z)$

(1) Estimation of $\text{RHS}(\Psi_0, \frac{dp}{d\psi}, \frac{df^2}{d\psi})^{k+1}$ from $(\Psi^k, q(\rho), \frac{dp}{d\rho})$

(2) A ’standard’ non-linear GSE solver, to update the equilibrium: $\text{RHS}^{k+1} \rightarrow \Psi^{k+1}(R, Z)$

(involves “internal” Newton iterations)

Iterate (1)-(2) over $k$ until target $q$ is reached (“external” iteration)
External iterations, an example

$q$ profiles after two iterations

$q$ profiles after two iterations

$p = 0$ (seed)

$k = 6$

Target: $#147626$, $t=5s$

$k = 1$

$k = 2$

$k = 0$
Convergence and performance I

- Total Newton iterations at $k=5 = 26$

Graphs showing the relative error and the number of Newton iterations for different values of $k$.
RHS \( \Psi_0, \frac{dp}{d\psi}, \frac{df^2}{d\psi} \) \( k+1 \)
estimation from \( \Psi^k(R, Z), q(\rho) \) and \( p(\rho) \)

\[
\Psi_0 = \int_0^1 \frac{\Phi_b^2 \rho}{\pi q} \, d\rho, \quad \frac{\partial \psi}{\partial \rho} = \frac{\Phi_b^2 \rho}{\pi \Psi_0 q}, \quad \frac{dp}{d\rho} = \frac{dp}{d\psi} \frac{\partial \psi}{\partial \rho}, \quad V_\rho = \frac{\partial V}{\partial \rho}
\]

\[
\langle GSE \rangle_k \rightarrow \frac{\Psi_0^2}{V_\rho} \frac{\partial}{\partial \rho} \left[ \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle V_\rho \frac{\partial \psi}{\partial \rho} \right] = -\frac{dp}{d\psi} - \frac{\langle R^{-2} \rangle}{2} \frac{df^2}{d\psi},
\]

\[
\langle GSE \rangle_{k+1} \rightarrow \frac{\Psi_0^2}{V_\rho} \frac{\partial}{\partial \rho} \left[ \left\langle \frac{|\nabla \rho|^2}{R^2} \right\rangle V_\rho \frac{\partial \psi}{\partial \rho} \right] = -\frac{dp}{d\psi} - \frac{\langle R^{-2} \rangle}{2} \frac{df^2}{d\psi},
\]

\[
\frac{df^2}{d\psi} = \alpha \frac{df^2}{d\psi} - C\frac{R_0}{\pi} \frac{I(\rho)}{\rho} \frac{\partial \alpha}{\partial \rho} - \frac{2}{\langle R^{-2} \rangle} \left( \frac{dp}{d\psi} - \alpha \frac{dp}{d\psi} \right), \quad \alpha = \frac{\Psi_0 q}{\Psi_0 q}
\]
Convergence and performance II

\[ F = R_0 B \phi / \rho \delta q / q < 1\% \]

1 Ext iter

1 Ext iter

\[ \text{Total Newton Iterations} = 1000 \]

\[ \text{Total Newton Iterations} = 650 \]

\[ \text{Relative Error} \]

\[ \text{t (s)} \]
Summary and perspective

- A new algorithm to include the equilibrium condition in control oriented transport simulations was developed, which is robust and reasonable efficient.

- The iterative algorithm has a physically intuitive basis and could be applied straightforwardly to existing eulerian GSE solvers and other applications.

- Moreover, the same principle can be extended to use different input data such as the radial dependence of the pitch-angle.

- Improvement of the efficiency in cases with significant changes in magnetic geometry must be addressed.

- Inclusion of the effect of the coils on the shape of the plasma (free boundary problem) in an appropriate manner is being studied.