Pfirsch-Tasso versus standard approaches in the plasma stability theory

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Abstract. The study is devoted to theoretical description of plasma stability in toroidal fusion systems with a resistive wall. Its aim is elimination of contradictions between different approaches and between theory and experiment. It is focused on the Pfirsch-Tasso approach originated from the paper published in 1971 (Nuclear Fusion, p. 259). The main relations have been given there without detailed proofs. Here, a missing chain of derivations is restored and earlier unknown constraints that restrict the applicability of the Pfirsch-Tasso energy principle are established. Its replacement is proposed. The new result is free from the constraints implicitly imposed in the Pfirsch-Tasso models. It eliminates the contradictions and can be used with any plasma model (not necessarily ideal) and for arbitrary perturbations. The proposed extensions allow applications for analysis of the rotational stabilization and optimization of the ITER scenarios.

1. Introduction

The main goal of this study is the revision of the Pfirsch-Tasso approach originating from the paper [1] where a famous theorem on MHD-instability of plasmas with resistive walls is presented (MHD: magnetohydrodynamics). It is based on the energy relation postulated in [1] and later applied in [2–4] as if already proved, though its derivation has been only briefly outlined. The paper [1] was published in 1971, but since then the proposed model and conclusions have never been analyzed, confirmed or corrected by independent researchers. This task is addressed here.

The original theorem says [1] that an MHD-unstable configuration with a dissipationless plasma surrounded by vacuum and possibly superconducting walls cannot be stabilized by introducing walls of finite electrical conductivity. In [1], a static plasma was implied, but an extension in [3] led to the enforced conclusion that in the absence of dissipation in the plasma such as viscosity, it is expected that the flow cannot stabilize the system. According to these statements, the wall stabilization must be negligibly weak in tokamaks. Actually, it is quite strong in experiments which makes it a viable concept for ITER [5] and JT-60SA [6]. This discrepancy is complemented by disagreements with a number of theoretical studies on the plasma rotation effect on the stability. Such contradictions are not rare [7, 8], but still remain unresolved. This necessitates a revision of the existing theory. Our study is focused on the Pfirsch-Tasso approach [1–4]. The purpose is to reveal its inherent limitations and eliminate them by proper improvements.

2. The model and formulation of the problem

A toroidal plasma separated from the vacuum vessel by a vacuum gap is considered. The important differences from the classical stability approaches [9–12] are that the vessel wall is not assumed ideal, and no particular assumptions on the plasma dynamics are introduced at the start of derivations. The latter allows to couple the final result to any plasma model, as

explained in [8]. Alternatively, the remaining freedom can be used for incorporation of experimental data when theoretical elements are unconstrained or, possibly, unreliable. At such formulation, the plasma-wall electromagnetic interaction is treated before specifying the plasma model. Accordingly, the sequence of derivations is opposite to that in [9–12]: the emphasis is laid on establishing that part of the energy balance which represents the outer region with dissipation in the conducting structures (traditionally called 'wall').

We start the stability analysis from the linearized force-balance equation

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F} \tag{1}$$

supplemented by the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\mathbf{j} = \nabla \times \mathbf{B}, \tag{3}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \tag{4}$$

and the Ohm's law for the resistive wall

$$\mathbf{j} = \sigma \mathbf{E}. \tag{5}$$

These are the same equations as in [1–4] and in various models used in the resistive wall mode (RWM) stability studies [7, 8]. Here ρ_0 is the unperturbed plasma mass density, ξ is the displacement from equilibrium, t is the time, $\mathbf{F}$ is the force operator, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ with $\mathbf{B}_0$ the stationary equilibrium magnetic field ($\partial \mathbf{B}_0 / \partial t = 0$) and

$$\mathbf{b} \equiv \mathbf{B} - \mathbf{B}_0 = \nabla \times \mathbf{A} \tag{6}$$

the time-varying magnetic perturbation, $\mathbf{A}$ is the vector potential under the gauge condition $\nabla \cdot \mathbf{A} = 0$ adopted in [1–4], $\mathbf{j}$ is the current density, $\mathbf{E}$ is the electric field, and σ is the electric conductivity of the wall ($\sigma = 0$ in vacuum). The perturbations of electromagnetic quantities are related by

$$\tilde{\mathbf{j}} = \nabla \times \mathbf{b} = \nabla \times \nabla \times \mathbf{A}, \tag{7}$$

where, according to (4)–(6), the induced currents in the wall must be subject to

$$\tilde{\mathbf{j}} = -\sigma \dot{\mathbf{A}}. \tag{8}$$

In [1–4] it is stated that the consequence of these equations is

$$\frac{d}{dt}(K + W) = -(\sigma \dot{\mathbf{A}}, \dot{\mathbf{A}})_{or}, \tag{9}$$

where

$$K \equiv \frac{1}{2} \int_{pl} \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2 dV, \tag{10}$$

$$W \equiv -\frac{1}{2} \int_{pl} \xi \cdot \mathbf{F} dV + \frac{1}{2} (\mathbf{A}, \tilde{\mathbf{j}})_{or}, \tag{11}$$

the dot means the time derivative, the subscripts pl and or denote, respectively, the plasma and the region outer to the plasma, and, as in [1–3],

$$(\boldsymbol{\eta}, \boldsymbol{\mu})_{or} \equiv \int_{or} \boldsymbol{\eta} \cdot \boldsymbol{\mu} dV. \tag{12}$$

Next we discuss the features of (9). Then we perform the derivations to show that a direct consequence of (1)–(5) must differ from (9). A new result will be a generalization of (9).

3. Discussion of the Pfirsch-Tasso relation (9)

Equations (9)–(11) are equivalent to Eqs. (4) in [1] with two minor differences in the notations. Specifically, we introduced the perturbed current density $\tilde{\mathbf{j}}$ in (11) instead of $\nabla \times \nabla \times \mathbf{A}$ used in formulas in [1–3]. This is done to explicitly show that W in the Pfirsch-Tasso model depends on $\tilde{\mathbf{j}}$ in the wall, while the classical stability theory [9–12] was developed for the case with $(\mathbf{A}, \tilde{\mathbf{j}})_{or} = 0$, i.e. either without wall, when $\tilde{\mathbf{j}} = 0$, or in the ideal-wall limit, when $\tilde{\mathbf{j}}$ becomes a surface current. Also, our $\mathbf{F}$ corresponds to $-\mathbf{F}$ in [1,2].

The first term in W defined by (11) coincides with the "potential energy" (denoted by W in [10] or δW in [9]) in the standard MHD stability models with an ideal wall. It also clearly resembles δW in [11] with $\boldsymbol{\xi}^* \cdot \mathbf{F}$ instead of $\boldsymbol{\xi} \cdot \mathbf{F}$ in (11).

However, at $\tilde{\mathbf{j}} \neq 0$ in the wall, the second term in (11) makes thus constructed W different from the energy functionals used in the traditional stability approaches.

Possible advantages of introducing W by (11) have never been demonstrated, and a full derivation of (9) with all theoretical details has never been attempted. Maybe, these are the reasons why the applications based on (9) are quite rare, see [2–4] and, for comparison, the reviews of alternative approaches in [7] and [8].

Equation (9) looks simple and elegant, except for the presence of the "resistive" terms in both W and the right-hand side. Another attractive feature is that Eq. (9) appears [1, 2] from (1) as straightforwardly derived by integrations shown in (11). It is implied in [1] and directly stated in [2] that $\mathbf{F}$ in (1) is the well-known stability operator of ideal MHD, but definition (11) does not rely on any specific property of $\mathbf{F}$. From explanations of the actions in [1–3] and from the result (9) itself it seems that $\mathbf{F}$ can be arbitrary. This idea is supported by the discussion in [3] where the same relation (9) is proposed for the plasma with stationary flows, when $\mathbf{F}$ becomes the Frieman-Rotenberg [12] operator. In [2] it is used assuming $\partial \sigma / \partial t \neq 0$.

In the standard stability theory [9–11] for the plasma subject to the ideal MHD constraints, the proof of (9) at $(\sigma \dot{\mathbf{A}}, \dot{\mathbf{A}})_{or} = 0$ (no wall or ideal wall) is essentially based on self-adjointness of the force operator $\mathbf{F}$. If we start derivations from (1) and follow the well developed procedure, we find that (9) cannot be obtained without this property. If it has been implicitly used on the way to (9) in [1–4], we also need a proof that neither the wall resistivity, nor the plasma flow would spoil it. This was not discussed in [1–4], even though the negative answer for the Frieman-Rotenberg operator is well known [12, 13].

The latter arguments could be of less importance if the 'kinetic energy' K in (9) would be neglected. Disregard of K is acceptable in the theory of relatively slow modes such as conventional RWMs [5, 7, 8]. Then the terms with $(\mathbf{A}, \tilde{\mathbf{j}})_{or}$ in (11) and the right-hand side in (9) become essential. Therefore, as a first step, we have to analyze them and assess whether they adequately reproduce the resistive wall effects in (9).

4. Derivations

In the pioneering Pfirsch-Tasso paper [1], the modified energy principle (9) with a resistive wall is just postulated: it appears as Eq. (4) there without derivations. Later this relation from [1] has been used in [2–4] as if already established. No mathematical arguments have been presented to justify applications of (9) for the cases with time-dependent conductivity of the wall in [2], stationary plasma flows in [3] and plasma stability control by external coils in [4].

The only explanation is given in [3], where the transition from (1) to (9) is described as taking the scalar product of Eq. (1) with $\partial \xi / \partial t$ and integrating over the plasma volume plus similar actions with the Ohm's law (8) for conductors in the outer region, multiplied by $\dot{\mathbf{A}}$. However, one can easily see that these actions do not lead to (9).

To get a correct replacement of (9), one can follow the same route as in the classical MHD stability theory [9–11] and incorporate the resistive wall as described, for example, in [8]. With such theoretical footing, the mathematics is not difficult. An important point is that we need a result in the form allowing easy comparison with the Pfirsch-Tasso relation (9).

The latter and the fact that K must be small for RWMs imply that, from three quantities in (9), we should consider W the most important and find a way that would yield the combination (11). The presence of $\xi \cdot \mathbf{F}$ in (11) suggests that we have to start by multiplying (1) by ξ (instead of $\partial \xi / \partial t$ proposed in [3]) and integrating afterwards over the plasma volume. The result

$$\int_{pl} \xi \cdot \left(\rho_0 \frac{\partial^2 \xi}{\partial t^2} - \mathbf{F} \right) dV = 0 \tag{13}$$

is easily transformed into

$$K + W = \frac{1}{2} \int_{pl} \rho_0 \left[\left(\frac{\partial \xi}{\partial t} \right)^2 - \xi \cdot \frac{\partial^2 \xi}{\partial t^2} \right] dV + \frac{1}{2} (\mathbf{A}, \tilde{\mathbf{j}})_{or}, \tag{14}$$

where definitions (10) and (11) have been used. Then after taking the time derivative we obtain (at $\partial \rho_0 / \partial t = 0$)

$$\frac{d}{dt} (K + W) = I_{pl} + \frac{1}{2} \frac{d}{dt} (\mathbf{A}, \tilde{\mathbf{j}})_{or} \tag{15}$$

with

$$I_{pl} \equiv \frac{1}{2} \int_{pl} \rho_0 \left(\dot{\xi} \cdot \frac{\partial^2 \xi}{\partial t^2} - \xi \cdot \frac{\partial^3 \xi}{\partial t^3} \right) dV . \tag{16}$$

Since in the outer region with (8) we have $\dot{\mathbf{A}} \cdot \tilde{\mathbf{j}} = -\sigma \dot{\mathbf{A}}^2$ and, accordingly,

$$\frac{\partial}{\partial t} (\mathbf{A} \cdot \tilde{\mathbf{j}}) = -2\sigma \dot{\mathbf{A}}^2 + \mathbf{A} \cdot \frac{\partial \tilde{\mathbf{j}}}{\partial t} - \dot{\mathbf{A}} \cdot \tilde{\mathbf{j}}, \tag{17}$$

equation (15) can be cast in the form

$$\frac{d}{dt} (K + W) = -(\sigma \dot{\mathbf{A}}, \dot{\mathbf{A}})_{or} + I_{pl} + I_{or} \tag{18}$$

with

$$I_{or} \equiv \frac{1}{2} \int_{or} (\mathbf{A} \cdot \frac{\partial \tilde{\mathbf{j}}}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} \cdot \tilde{\mathbf{j}}) dV = \frac{1}{2} \int_{wall} \sigma (\dot{\mathbf{A}}^2 - \mathbf{A} \cdot \ddot{\mathbf{A}}) dV - \frac{1}{2} \int_{wall} \mathbf{A} \cdot \dot{\mathbf{A}} \frac{\partial \sigma}{\partial t} dV . \tag{19}$$

The last equality here is obtained with (8). The term with $\partial \sigma / \partial t$ is retained in order to cover the case with time-dependent wall resistivity discussed in [2]. We only note that such term is absent in [2]. Below we treat σ as a constant, $\partial \sigma / \partial t = 0$, which is natural in practical tasks.

The operations leading from (13) to (18) do not rely on any property of $\mathbf{F}$. Therefore, equation (18) must be valid for any $\mathbf{F}$ in (1). Note that, to get the functional W defined by (11), we simply add $(\mathbf{A}, \tilde{\mathbf{j}})_{or}$ to the both sides of (13). The choice of this particular quantity in [1–3] has not been unexplained. Such transformation of (13) can be done with anything else instead of $(\mathbf{A}, \tilde{\mathbf{j}})_{or}$, if we find it necessary or useful. This freedom can be potentially used to

gain a better form of the final relation. For example, if the target equality must be precisely (9) with only $-(\sigma\dot{\mathbf{A}}, \dot{\mathbf{A}})_{or}$ on the right-hand side, we have to replace $(\mathbf{A}, \tilde{\mathbf{j}})_{or}$ in (11) by

$$(\mathbf{A} \cdot \tilde{\mathbf{j}})_{or} - 2\int I_{or}dt = -2\int(\sigma\dot{\mathbf{A}}, \dot{\mathbf{A}})_{or}dt. \tag{20}$$

These are purely theoretical arguments. Next we have to find the implications.

5. Comparison of (9) and (18)

Our equation (18) looks closely similar to the key relation (9) of the Pfirsch-Tasso approach [1–4], but contains two additional terms, I_{pl} and I_{or}. It is clear that $I_{pl} = I_{or} = 0$ for ξ and $\mathbf{A} \propto \exp(\gamma t)$. Then equations (18) and (9) become identical. However, for perturbations like

$$x = x_1 \exp(\gamma_1 t) + x_2 \exp(\gamma_2 t) \tag{21}$$

we would obtain $I_{or} \neq 0$ because

$$x\ddot{x} - \dot{x}^2 = x_1 x_2 (\gamma_1 - \gamma_2)^2 \exp(\gamma_1 + \gamma_2)t. \tag{22}$$

This proves that equations (18) and (9) may be different.

Before elaborating this point, let us note that I_{pl} defined by (16) must be small for slow modes such as RWMs [5, 7, 8]. Here the same arguments are applied as those validating the disregard of the kinetic energy K (or its equivalents) in theoretical studies of RWMs [7, 8]. Therefore, when the wall resistivity affects the plasma stability, I_{or} should be considered the main additional term in (18) compared to (9).

A complete theoretical analysis of the functional I_{pl} should be done by using the expression

$$I_{pl} \equiv \frac{1}{2} \int_{pl} (\dot{\xi} \cdot \mathbf{F} - \xi \cdot \dot{\mathbf{F}})dV \tag{23}$$

obtained from (16) with (1). It is known that, irrespective of the time dependence of ξ, such $I_{pl} = 0$ in the standard ideal MHD stability theory [9–11]. Precisely, this is true for the ideal plasma perturbed from the static equilibrium, if, in addition, $\sigma = 0$ (no wall) or $\partial\mathbf{A}/\partial t = 0$ in the wall (ideally conducting wall). Then $\mathbf{F} = \mathbf{F}_{id}(\xi)$ with

$$\mathbf{F}_{id} \equiv -\nabla\tilde{p} + \mathbf{j}_0 \times \mathbf{b} + \tilde{\mathbf{j}} \times \mathbf{B}_0, \tag{24}$$

$\tilde{p} = p - p_0$ the pressure perturbation and $\mathbf{j}_0 = \nabla \times \mathbf{B}_0$ the equilibrium current. The property $I_{pl} = 0$ or, more general,

$$\int_{pl} \{\boldsymbol{\eta} \cdot \mathbf{F}_{id}(\xi) - \xi \cdot \mathbf{F}_{id}(\boldsymbol{\eta})\}dV = 0, \tag{25}$$

where both ξ and $\boldsymbol{\eta}$ are the solutions of (1) with $\mathbf{F} = \mathbf{F}_{id}$ under the ideal-wall boundary conditions, plays a special role in the stability studies based on the ideal MHD model for a static plasma. The proofs of (25) in [9–11] cannot be directly extended to the plasmas with stationary flows, but the non-self-adjointness of the force operator in this case can be elegantly accommodated as described in [13], though again under the ideal-wall assumption.

Here we assume the wall resistive and up to now do not impose any restriction on $\mathbf{F}$. It can differ from $\mathbf{F}_{id}$ defined by (24) even for the ideal plasma if the plasma rotation is allowed [3,

12, 13]. Therefore, one cannot always expect that $I_{pl} = 0$, though it must be a reasonable approximation for slow modes at any $\mathbf{F} \neq \mathbf{F}_{id}$, see equation (16).

Situation is different with I_{or}. Consider a rotating mode with

$$\mathbf{A} = \mathbf{A}_0 \cos n(\zeta - \omega t), \tag{26}$$

where ω is the angular frequency of the mode rotation along the toroidal angle ζ, n is the toroidal wave number, and $\mathbf{A}_0$ is the amplitude independent of ζ and t. This is clearly a particular case of (21) with $\gamma_1 = -\gamma_2 = in\omega$. Then we obtain for an axisymmetric wall:

$$(\sigma\dot{\mathbf{A}}, \dot{\mathbf{A}})_{or} = I_{or} = \frac{n^2\omega^2}{2}(\sigma\mathbf{A}_0, \mathbf{A}_0)_{or}. \tag{27}$$

This result proves that I_{or} can give an essential contribution into the right-hand side of (18). Its absence in the Pfirsch-Tasso expression (9) reduces the applicability of the latter to analysis of the modes with $\exp(\gamma t)$ dependence only. This has not been explained in [1–4]. On the contrary, (9) is always presented [1–4] as a general relation without any mentioning of its inherent limitations.

Note that (27) is equivalent to $(\mathbf{A} \cdot \tilde{\mathbf{j}})_{or} = 0$, though both $\mathbf{A}$ and $\tilde{\mathbf{j}}$ are nonzero. We also obtain $I_{pl} = 0$ for the rotating perturbations described by (26). Then equation (18) reduces to

$$\frac{d}{dt}(K + W) = 0, \tag{28}$$

while the Pfirsch-Tasso energy relation (9) gives us an essentially different result:

$$\frac{d}{dt}(K + W) = -\frac{n^2\omega^2}{2}(\sigma\mathbf{A}_0, \mathbf{A}_0)_{or} < 0. \tag{29}$$

The former is compatible with (26) and $\partial\mathbf{A}_0/\partial t = 0$, while the latter is not. In other words, equation (29) does not allow existence of a steady state with a rotating mode like (26).

Such modes, however, often appear in tokamaks [5, 7, 14]. An excellent example is the edge harmonic oscillation (EHO), which is a property of the quiescent H-mode (QH-mode) observed on several tokamaks [15], see also reviews in [8, 16]. The EHOs are registered by the magnetic probes near the wall [15]. If so, their interaction with the wall must provide $(\sigma\mathbf{A}_0, \mathbf{A}_0)_{or} \neq 0$, which is an essential part in the discussion based on (26) and (27).

We can conclude that the most important difference between (9) and (18) comes from I_{or} in (18). This term is absent in (9) and it never appeared in the Pfirsch-Tasso approach. Perhaps, some assumptions (like $\exp(\gamma t)$ dependence of perturbations) have been implicitly involved, but this fact has not been explained in [1–4]. According to (8) and (19), for perturbations with exponential growth/decay we get $I_{or} = 0$. Then the Pfirsch-Tasso relation (9) must be valid. Applicability of (9) in other cases (with nonexponential, e.g., algebraic growth rate mentioned in [2]) remains an open question. We proved that it is certainly incorrect for the rotating perturbations like (26). This is sufficient for disproof of the conclusion of [3] that the flow cannot stabilize the system. At least, it can be called unproved in [3].

From theoretical viewpoint, the above demonstration of non-applicability of (9) for perturbations described by (26) could be a solid argument if (26) would be indeed a solution

of (1). Here we suggest that it must be so because of EHOs and other rotating perturbations observed in experiments. If some force operator $\mathbf{F}$ in (1) does not allow such solutions, it should be replaced by something else. Let us remind that (18) is obtained without specifying $\mathbf{F}$, and the origin and use of (9) in [1–3] seems to be free from limitations on $\mathbf{F}$ that appear as a parameter in (1). That is why we are not restricted by the conventional ideal MHD.

6.   The Pfirsch-Tasso energy functional

Assume that $\mathbf{F} = \mathbf{F}_{id} + \mathbf{F}_{non}$, where $\mathbf{F}_{non}$ describes the difference of $\mathbf{F}$ from the ideal MHD force operator $\mathbf{F}_{id}$ given by (24). Then

$$-\frac{1}{2}\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}dV = W_p + W_s + W_{non} + \frac{1}{2}\oint_{pl}(\mathbf{b}_e\cdot\mathbf{B}_{0e})\boldsymbol{\xi}\cdot d\mathbf{S}_{pl}\,, \tag{30}$$

where

$$2W_p \equiv \int_{pl}[\mathbf{b}\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) + \boldsymbol{\xi}\cdot(\mathbf{b}\times\mathbf{j}_0) - \widetilde{p}\nabla\cdot\boldsymbol{\xi}]dV\,, \tag{31}$$

$$2W_s \equiv \oint_{pl}[\mathbf{b}_i\times(\boldsymbol{\xi}\times\mathbf{B}_{0i}) - (\mathbf{b}_e\cdot\mathbf{B}_{0e})\boldsymbol{\xi}]\cdot d\mathbf{S}_{pl} + \oint_{pl}\widetilde{p}\boldsymbol{\xi}\cdot d\mathbf{S}_{pl}\,, \tag{32}$$

$$W_{non} \equiv -\frac{1}{2}\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_{non}dV\,, \tag{33}$$

the subscripts i and e denote, respectively, the inner and outer regions with respect to the plasma boundary S_{pl}. Note that (31) and (32) are the standard definitions of the energy functionals in the ideal MHD stability theory [9–11]. With $\mathbf{b} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)$ and $\widetilde{p} = -\boldsymbol{\xi}\cdot\nabla p_0 - \Gamma p_0\,\mathrm{div}\,\boldsymbol{\xi}$, which is valid for the ideal plasma, they give the well-known expressions for W_p and W_s, see [9–11]. However, for a non-ideal plasma with other relations for $\mathbf{b}$ and $\widetilde{p}$ the results will different.

Introducing some 'vector-potential' $\mathbf{q}$ outside the plasma subject to the boundary condition

$$\mathbf{n}_{pl}\times\mathbf{q} \equiv -(\boldsymbol{\xi}\cdot\mathbf{n}_{pl})\mathbf{B}_{0e} \tag{34}$$

on the plasma surface, we can transform the last term in (30) by the formula

$$\oint_{pl}(\mathbf{b}_e\cdot\mathbf{B}_{0e})\boldsymbol{\xi}\cdot d\mathbf{S}_{pl} = 2W_m - \int_{wall}\mathbf{q}\cdot\widetilde{\mathbf{j}}dV\,, \tag{35}$$

where

$$W_m \equiv \frac{1}{2}\int_{out}\mathbf{b}\cdot(\nabla\times\mathbf{q})dV\,. \tag{36}$$

This allows to cast (30) in the form

$$-\frac{1}{2}\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}dV + \frac{1}{2}\int_{wall}\mathbf{q}\cdot\widetilde{\mathbf{j}}dV = W_p + W_s + W_m + W_{non}. \tag{37}$$

With $\mathbf{q} = \mathbf{A}$, equation (34) reproduces the standard matching [9–11] at the plasma boundary, when the plasma does not rotate and is treated as ideally conducting. In this case, substitution $\mathbf{q} = \mathbf{A}$ turns W_m into the perturbation-produced magnetic energy in the space outside the plasma. This space would be just the plasma-wall gap, if the wall is also ideal. If not, it extends beyond the wall. The implications have been discussed in [8].

At $\mathbf{q} = \mathbf{A}$, the left-hand side of (37) will be exactly the Pfirsch-Tasso W defined by (11), and the right-hand side of (37) will give it in terms of customary functionals W_α. Therefore, definition (11) is convenient for the ideal non-rotating plasma, when it reduces to rhs of (37). If the plasma is non-ideal and $\mathbf{q} \neq \mathbf{A}$ (when we cannot substitute $\mathbf{A}$ instead of $\mathbf{q}$ in equation (34)), the functional (11) will contain an additional contribution depending on $\mathbf{q} - \mathbf{A}$.

7. Conclusion

Essential, previously unknown limitations of the Pfirsch-Tasso approach [1–4] are established. The contradictions between the models of the rotational stabilization [7, 8] and the theorems in [1–4] must be a consequence of the inherent restrictions of the method in [1–4]. The proposed replacement of the Pfirsch-Tasso energy principle is free from the constraints imposed in [1–4] and can be used with any plasma model (not necessarily ideal, as in [1–4]) and with arbitrary time-dependence of perturbations. This allows applications for the cases of practical interest such as feedback stabilization of RWMs, analysis of the rotational stabilization and optimization of the ITER and JT-60SA scenarios.

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