CONFERENCE PRE-PRINT

EFFECTS OF FINITE ION TEMPERATURE AND ITS GRADIENT ON HASEGAWA-MIMA EQUATION AND ZONAL FLOW GENERATION

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Abstract

The effects of finite ion temperature and its gradient on the Hasegawa–Mima equation and zonal flow generation are investigated. The standard Hasegawa–Mima equation, based on the cold-ion assumption, becomes inadequate in high-temperature fusion core plasmas. A modified model incorporating finite ion temperature, ion diamagnetic drift, and finite Larmor radius effects is derived from gyro-fluid equations. It is demonstrated that potential vorticity is local conserved rather than a Lagrangian invariant, and potential enstrophy is not conserved anymore, for uniform plasmas. The zonal flow evolution equation is extended with additional contributions from ion diamagnetic Reynolds stress. Moreover, theoretical analysis shows that finite ion temperature and its gradient can enhance zonal flow generation by modifying the turbulent pseudo-momentum allocation. These results improve the understanding of drift-wave–zonal-flow interactions in magnetically confined fusion plasmas.

1. INTRODUCTION

Micro-turbulence and associated anomalous transport, which can be regulated by zonal flow (ZF) are important topics for the achievement of ignition conditions in future tokamak fusion reactor. The Hasegawa-Mima (H-M) equation, renowned for its simplicity and robust conservation properties [1], has been widely adopted to study drift-wave (DW) and ZF systems. However, its fundamental assumption of cold ions becomes invalid in high-temperature core plasmas of magnetic confinement fusion devices, necessitating modifications to incorporate effects of finite ion temperature and its gradient.

It is widely recognized that, for DW turbulence, the cross-correlation of the radial and poloidal components of the electric drift fluctuations $\tilde{v}_{Er}$ and $\tilde{v}_{E\theta}$, i.e., Reynolds stress is the drive of ZF. However, for the ion temperature gradient (ITG) turbulence, it is proposed that fluctuations of the diamagnetic flow $\tilde{v}_{*r}$ can also correlate with $\tilde{v}_{E\theta}$ to contribute the diamagnetic Reynolds stress and drive ZF [2, 3], which has been verified by GYSELA gyrokinetic simulation [4]. To account for ion pressure fluctuations, the standard H-M equation can be extended by relaxing the cold-ion approximation, which typically neglects finite ion temperature effects. Moreover, the potential vorticity (PV), total energy density and potential enstrophy (PE) of the standard H-M equation are conserved [1]. Therefore, discussion on whether these quantities are still conserved in extended H-M system should be interesting.

Based on the PV conservation in 2D drift-wave system, [5] derives a zonal momentum theorem, where stationary turbulence cannot excite a ZF in the absence of particle flux, dissipation and transport of PE. Later, this momentum theorem is extended to 3D coupled DW-ion acoustic wave (IAW) system [6] and to non-uniform magnetic field case [7]. In DW-IAW coupled system, conservation of PV is broken by fluctuating parallel flow compressibility, and the coupling of drift waves and ion acoustic waves can excite ZF even in the absence of a driving force and a PE flux. Thus, how is the momentum theorem modified in the extended H-M system with finite ion temperature is worth investigation.

In this work, we study the modification of finite ion temperature and its gradient to the H-M equation and the ZF generation via zonal momentum. Starting from gyro-fluid equations [8], a modified model incorporating finite ion temperature, ion diamagnetic drift, and finite Larmor radius effects is derived. It is found that the PV is only locally conserved rather than a Lagrangian invariant, and the PE is not conserved either for uniform plasmas. The ZF evolution equation is also extended with additional contributions from ion diamagnetic Reynolds stress. Moreover, theoretical analysis of the modified zonal momentum theorem shows that finite ion temperature and its gradient can enhance ZF generation by modifying the turbulent pseudo-momentum allocation.

The remainder of this paper is organized as follows. In section 2, we present the equations of PV and PE based on the gyro-fluid equation, and the relevant conservation properties are discussed. Then, we derive the zonal momentum theorem in section 3. Finally, we summarize our work in section 4.

2. CONSERVATION PROPERTIES

We start from the gyro-fluid equations in slab geometry [8]

$$\frac{\partial}{\partial t}\tilde{q} + \tilde{\boldsymbol{v}}_E \cdot \nabla\tilde{q} - \tau_{ie}\tilde{\boldsymbol{v}}_* \cdot \nabla\left(\nabla_\perp^2\tilde{\phi}\right) - \tau_{ie}\hat{\boldsymbol{b}} \times \left(\nabla\nabla_{\perp i}\tilde{p}_\perp\right) \cdot \left(\nabla\nabla_{\perp i}\tilde{\phi}\right) = \omega_{*e}[1 + \tau_{ie}(1+\eta_i)\nabla_\perp^2]\tilde{\phi}. \tag{1}$$

Here, $\tilde{q} = \tilde{\phi} - \nabla_\perp^2\tilde{\phi}$ is the PV fluctuation. The standard normalizations are used as follows: the electric potential fluctuation $\tilde{\phi} \equiv \frac{e\delta\phi}{T_{e0}}$, ion perpendicular pressure fluctuation $\tilde{p}_\perp \equiv \frac{\delta P_{\perp i}}{P_{i0}}$ with $P_{i0} = n_0 T_{i0}$. The ion pressure fluctuation comes from the finite Larmor radius effects, since the polarization density is $\nabla_\perp^2\tilde{\phi} - \tau_{ie}\frac{1}{2}\nabla_\perp^2\tilde{p}_\perp$. Time scale is normalized by Ω_i with $\Omega_i = eB/(m_i c)$ being the ion gyrofrequency, spatial scale by ρ_s with $\rho_s = \sqrt{\frac{T_{e0}}{m_i}}/\Omega_i$ being the ion gyroradius at acoustic velocity, $\tilde{\boldsymbol{v}}_E = \hat{\boldsymbol{b}} \times \nabla\tilde{\phi}$ and $\tilde{\boldsymbol{v}}_* = \hat{\boldsymbol{b}} \times \nabla\tilde{p}_\perp$ are the fluctuating $\boldsymbol{E} \times \boldsymbol{B}$ and ion diamagnetic drift velocities, respectively, $\omega_{*e} = -k_y\left(\frac{\partial}{\partial r}\ln n_0\right)$ is the electron diamagnetic frequency, $\eta_i = \left(\frac{\partial}{\partial r}\ln T_{i0}\right) / \left(\frac{\partial}{\partial r}\ln n_0\right)$ and $\tau_{ie} = \frac{T_{i0}}{T_{e0}}$ is the ratio of ion temperature to electron temperature. For cold ion approximation, i.e., $\tau_{ie} \to 0$, all the terms proportional to $\tilde{p}_\perp$ and equilibrium ion diamagnetic drift in equation (1) vanish, and the standard H-M equation will be reduced.

For uniform plasmas, equation (1) is reduced to

$$\frac{\partial}{\partial t}\tilde{q} + \tilde{\boldsymbol{v}}_E \cdot \nabla\tilde{q} - \tau_{ie}\tilde{\boldsymbol{v}}_* \cdot \nabla\left(\nabla_\perp^2\tilde{\phi}\right) - \tau_{ie}\hat{\boldsymbol{b}} \times (\nabla\nabla_{\perp i}\tilde{p}_\perp) \cdot \left(\nabla\nabla_{\perp i}\tilde{\phi}\right) = 0. \tag{2a}$$

In standard H-M equation where PV is a Lagrangian invariant, i.e., $\frac{d}{dt}\tilde{q} = 0$ with $\frac{d}{dt} = \frac{\partial}{\partial t} + \tilde{\boldsymbol{v}}_E \cdot \nabla$. However, from equation (2a), we do not have $\frac{d}{dt}\tilde{q} = 0$ anymore, even with modified convection velocity. It is ion pressure fluctuation $\tilde{p}_\perp$ breaks the Lagrangian invariant of PV. Using the Poisson bracket $\{f, g\} = \left(\hat{b} \times \nabla f\right) \cdot \nabla g$, equation (2a) can be rewritten as

$$\frac{\partial}{\partial t}\tilde{q} - \nabla_{\perp i}\left\{\tilde{\phi} + \tau_{ie}\tilde{p}_\perp, \nabla_{\perp i}\tilde{\phi}\right\} = 0. \tag{2b}$$

Fortunately, we can see that the PV is still locally conserved.

Multiplying equation (1) by $\tilde{q}$, we can obtain the evolution equation of PE

$$\frac{\partial}{\partial t}\langle\frac{\tilde{q}^2}{2}\rangle + \frac{\partial}{\partial x}\langle\tilde{v}_{Ex}\frac{\tilde{q}^2}{2}\rangle + \tau_{ie}\frac{\partial}{\partial x}\langle\tilde{v}_{*x}\frac{1}{2}\left[\left(\nabla_\perp^2\tilde{\phi}\right)^2 + \left(\nabla_\perp\tilde{\phi}\right)^2\right]\rangle - \tau_{ie}\frac{\partial}{\partial x}\langle\tilde{\phi}\left(\tilde{v}_{*y}\frac{\partial}{\partial y} + \tilde{v}_{*x}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\tilde{\phi}\rangle$$
$$+\langle\nabla_\perp^2\tilde{\phi}\left\{\tau_{ie}\nabla_{\perp i}\tilde{p}_\perp, \nabla_{\perp i}\tilde{\phi}\right\}\rangle = -\left(\frac{\partial}{\partial r}\ln n_0\right)\left[1 + \tau_{ie}(1 + \eta_i)\right]\langle\tilde{v}_{Ex}\tilde{q}\rangle. \tag{3}$$

In addition to the PE flux due to $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity, there are more flux terms-the second and the third terms on the left-hand side (LHS) of equation (3)-resulting from fluctuating diamagnetic drift velocity. Moreover, the last term on the LHS of Equation (3), contributed by the ion pressure fluctuations, breaks the Lagrangian invariant of PE. Therefore, the PE is not conserved for uniform plasmas either within the extended H-M system.

3. ZONAL MOMENTUM THEOREM

In this section, we derive the modified zonal momentum theorem for extended H-M system. Taking flux average of equation (2), integrating in *x* direction and adding the damping, ZF evolution equation can be obtained as

$$\frac{\partial}{\partial t}\langle v_y\rangle + \frac{\partial}{\partial x}\langle(\tilde{v}_{Ex} + \tilde{v}_{*x})\tilde{v}_{Ey}\rangle = -\nu\langle v_y\rangle. \tag{4}$$

In addition to the usual Reynolds stress force, the ion diamagnetic Reynolds stress $\langle\tilde{v}_{*x}\tilde{v}_{Ey}\rangle$ can also contribute ZF drive. This is consistent with previous works [2, 3, 4]

Then, dividing equation (3) by a factor $\alpha = \left(\frac{\partial}{\partial r}\ln n_0\right)\left[1 + \tau_{ie}(1 + \eta_i)\right]$ and adding the ZF evolution equation (4), we can obtain the total zonal momentum equation [3, 4] modified by the effects of FLR and the fluctuating ion diamagnetic drift velocity

$$\frac{\partial}{\partial t}\left[\langle v_y\rangle + \langle\frac{\tilde{q}^2}{2\alpha}\rangle\right] = -\nu\langle v_y\rangle - \frac{\partial}{\partial x}\langle\tilde{v}_{Ex}\frac{\tilde{q}^2}{2\alpha}\rangle - \frac{\partial}{\partial x}\langle\tilde{v}_{*x}\,\tilde{v}_{Ey}\rangle + \frac{\partial}{\partial x}\langle\frac{\tilde{v}_{Ex}\tilde{v}_{*y}\tilde{v}_{Ey}}{\alpha}\rangle$$
$$-\frac{\partial}{\partial x}\langle\tilde{v}_{*x}\frac{\left(\nabla_\perp^2\tilde{\phi}\right)^2 + \left(\nabla_\perp\tilde{\phi}\right)^2 + 2\tilde{v}_{*y}^2}{2\alpha}\rangle + \frac{\partial^2}{\partial x^2}\langle\tilde{v}_{*x}\frac{\partial^2}{\partial x^2}\left(\frac{\tilde{\phi}^2}{2\alpha}\right)\rangle \tag{5}$$

It is found that the ion diamagnetic drift contributes additional ZF drive via its effects on Reynolds stress and turbulence spreading. Moreover, under fixed PE intensity, finite ion temperature and its gradient reduce the allocation of turbulent pseudo-momentum in the total zonal momentum as compared to the 2D DW case, thereby enhancing the ZF generation. This may be important for the confinement improvement of core plasmas where cold-ion approximation is not valid.

4. SUMMARY

This work presents a comprehensive analysis of how finite ion temperature and its radial gradient modify the foundational H-M equation and impact ZF generation in magnetically confined plasmas. It is found that the conservation properties of PV and PE are broken, and the ZF drive is increased due to additional drive from the ion pressure fluctuations and the reduction of turbulent pseudo-momentum. The standard H-M model, which relies on the cold-ion assumption, is inadequate for describing high-temperature core plasma physics. To address this, we derive an extended gyro-fluid model that self-consistently incorporates key effects of finite ion temperature and its gradient, and the fluctuating ion diamagnetic drift induced by ion pressure fluctuations. The conservation laws inherent to the original H-M equation are profoundly altered in this more general framework. It is shown that while PV remains locally conserved, it is not a Lagrangian invariant anymore. Furthermore, PE is no longer a conserved quantity, even in uniform plasmas, due to the presence of additional flux terms and a source term arising from ion pressure fluctuations.

A key advancement is the derivation of a modified ZF evolution equation, which now includes a drive term from the ion diamagnetic Reynolds stress, consistent with previous gyrokinetic simulation results. This leads to a generalized zonal momentum theorem that accounts for the new physics. The theoretical analysis demonstrates that finite ion temperature and its gradient can enhance ZF generation by modifying the turbulent pseudo-momentum allocation within the total momentum balance. This enhancement mechanism is potentially significant for improving confinement in fusion reactor cores, where cold-ion models are inapplicable. The results provide a more complete and physically accurate framework for understanding the nonlinear interplay between drift-wave turbulence and ZFs in high-temperature plasmas.

ACKNOWLEDGEMENTS

This work was supported by the National Key R&D Program of China No. 2024YFE03060001, the National Natural Science Foundation of China under Grant Nos. 12275097 and 12275096, and Hubei International Science and Technology Cooperation Projects (No. 2022EHB003). One of the authors, Lu Wang, would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program "Anti-diffusive dynamics: from sub-cellular to astrophysical scales," where part of this work on this paper was undertaken. This work was also supported by EPSRC Grant No. EP/R014604/1.

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