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NEOCLASSICAL THEORY ON LOW FREQUENCY DRIFT ALFVÉN WAVES

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Abstract

In this work, we developed kinetic models based on general fishbone-like dispersion relations. Firstly, by disregarding ion orbit width and approximating the magnetic geometry as circular, we introduce a simplified model that fully incorporates neoclassical effects by considering full circulating/trapped particles. Then, by considering the limit of ions being well-circulating or deeply trapped, the results directly revert to those observed in earlier theoretical studies. Finally, the accumulation point frequency of Beta-induced Alfvén Eigenmode is calculated in the fluid limit.

1. INTRODUCTION

Alfvén waves and energetic particles, resulted from fusion reaction and auxiliary heating, are crucial to the performance of Tokamak devices. The theoretical research on low frequency drift Alfvén waves (DAW) is based on the general fishbone-like dispersion relation (GFLDR)[1] and gyrokinetic theory[2]. Besides recovering diverse limits of the kinetic magnetohydrodynamic (MHD) energy principle, the GFLDR approach is also applicable to electromagnetic fluctuations, which exhibit a wide spectrum of spatial and temporal scales consistent with gyrokinetic descriptions of both the core and supra-thermal plasma components. Formally ,the GFLDR can be written as

$$i\Lambda = \delta W_f + \delta W_k, \qquad (1)$$

where Λ is the generalized inertia, and δW_f and δW_k are, respectively, fluid and kinetic potential energy of electromagnetic fluctuations. In previous researches, Λ is calculated with fluid and kinetic approaches[3].

In the original theoretical works[4] on DAW, ions considered in the kinetic analysis are assumed to be well circulating. Later on, the kinetic analysis was extended by including the deeply trapped ions and electrons[3]. Moreover, the researches mentioned above are all based on the $s-\alpha$ model in Tokamak plasmas with circular configuration. However, the neoclassical effects considering full circulating/trapped particles are not included in previous researches. Especially, the particles near circulating/trapped separatrix are not included in the previous theoretical models. In order to obtain a better understanding of experimental observations [5] and provide a more precise kinetic model for theoretical researches, we need to include full neoclassical effects without assuming well-circulating/deeply-trapped ions.

In this work, we present a kinetic model with $s-\alpha$ circular magnetic geometry and small ion orbit width. The generalized inertia with the modification of neoclassical effects can be calculated with appropriate circular geometry data. Furthermore, if we assume ions are well circulating or deeply trapped, the results go back to those of the previous researches[3]. Finally, the generalized inertia and the accumulation point frequency of Beta-induced Alfvén Eigenmode (BAE) can be calculated.

The rest of the paper is organized as follows. In Section 2, we present our kinetic model including guiding center motion in circular geometry, kinetic equation in ballooning space and governing equations. Section 3 is devoted to solve the governing equation in the inertial layer. The results in deep trap particle limit and fluid limit are presented in Section 4. Finally, the results are summarized in Section 5.

2. KINETIC MODEL

2.1. GUIDING CENTER MOTION

The guiding center motion of single particles can be given as

$$\dot{\mathbf{X}} = v_{\parallel}\mathbf{b} + \frac{v_{\parallel}}{\Omega_s}\nabla\times\left(\mathbf{b}v_{\parallel}\right), \quad (2)$$

where $v_{\parallel}$ is the velocity parallel to magnetic field, $\Omega_s = e_s B / m_s c$ is the gyro-frequency of the s-species and $\mathbf{b}$ is the unit vector of magnetic field. In this work, we will only consider the leading order solution without assuming well circulating and deeply trapped particles. For the circular up-down symmetric configuration, the magnetic field $B = B_0\left(1-\epsilon\cos\theta\right)$, where $\epsilon = r/R_0 \sim \delta^2$ is the inverse aspect ratio, r is the minor radius, R_0 is the major radius and δ is the small parameter. The guiding center motion is governed by[6]

$$\dot{\theta} = \frac{v_{\parallel}}{qR_0},$$

$$\dot{\zeta} - q\dot{\theta} = \frac{q}{2rR_0\Omega_s}\left(v^2 + v_{\parallel}^2\right)\cos\theta,$$

$$\dot{r} = \frac{v^2 + v_{\parallel}^2}{2R_0\Omega_s}\sin\theta,$$

where $v_{\parallel} = \sigma\sqrt{2\mathcal{E}\left(1-\lambda B/B_0\right)}$, $\lambda = \mu B_0/\mathcal{E}$, $v = \sqrt{2\mathcal{E}}$, $\sigma = \pm 1$ is the sign of parallel velocity, $\mathcal{E}$ is the particle energy per mass, θ is the poloidal angle, ζ is the toroidal angle and q is the safety factor.

2.2. GYROKINETIC EQUATION IN THE BALLOONING SPACE

Before going to ballooning space, we start the analysis from the gyro-kinetic equation in original space. The perturbed particle distribution function can be written as[2]

$$\delta f_s = \left(\frac{e}{m}\right)_s\left[\frac{\partial F_0}{\partial\mathcal{E}}\delta\phi - J_0\left(k_{\perp}\rho_{Ls}\right)\frac{QF_0}{\omega}\delta\psi e^{iL_k}\right]_s + \delta K_s e^{iL_{ks}}, \quad (3)$$

where $\delta\phi$ is perturbed electric potential, $k_{\perp}^2 = k^2 - \left(\mathbf{k}\cdot\mathbf{b}\right)^2$, $\delta\psi$ is related to parallel vector potential fluctuation $\delta A_{\parallel}$ with $\delta A_{\parallel} = -i\left(c/\omega\right)\mathbf{b}\cdot\nabla\delta\psi$, J_0 is the 0th Bessel function, $\rho_{Ls} = m_s c v_{\perp}/e_s B_0$ is the the Larmor radius, $QF_{0s} = \left[\omega\partial_{\mathcal{E}} + \Omega_s^{-1}\left(\mathbf{k}\times\mathbf{b}\right)\cdot\nabla\right]F_{0s}$, and $L_{ks} = \left(m_s c/e_s B_0\right)\left(\mathbf{k}\times\mathbf{b}\right)\cdot\mathbf{v}$. And the non-adiabatic part δK_s obeys the gyro-kinetic equation

$$\left(-i\omega + \dot{r}\partial_r + \dot{\theta}\partial_{\theta} + \dot{\zeta}\partial_{\zeta}\right)\delta K_s =$$
$$i\left(\frac{e}{m}\right)_s QF_{0s}\times\left[J_0\left(\delta\phi-\delta\psi\right) + \left(\frac{\omega_d}{\omega}\right)_s J_0\delta\psi + \frac{v_{\perp}}{k_{\perp}c}J_1\delta B_{\parallel}\right], \quad (4)$$

where J_1 is the 1st Bessel function, $\omega_{ds} = \left(\mathbf{k}\times\mathbf{b}\right)\cdot\left(\mu\nabla B_0 + v_{\parallel}^2\boldsymbol{\kappa}\right)/\Omega_s$, κ is the curvature of magnetic field line and $\delta B_{\parallel}$ is the perturbed parallel magnetic field. By considering the ballooning representation, we have[1]

$$f\left(\psi,\theta,\zeta\right) = e^{in\zeta}\sum_m e^{-im\theta}f_{m,n} = e^{in\zeta}\sum_m e^{-im\theta}\int e^{-i(nq-m)\vartheta}\hat{f}d\vartheta. \quad (5)$$

Then the gyro-kinetic equation can be rewritten as

$$\left[-i\omega + \dot{\theta}\partial_{\vartheta} + i\omega_{ds}\right]\delta\hat{K}_s = i\left(\frac{e}{m}\right)_s QF_{0s}\times\left[J_0\left(\delta\phi-\delta\psi\right) + \frac{\omega_{ds}}{\omega}J_0\delta\psi + \frac{v_{\perp}}{k_{\perp}c}J_1\delta B_{\parallel}\right], \quad (6)$$

where $\omega_{ds} = nq\left(v^2 + v_{\parallel}^2\right)\left(\cos\theta + s\vartheta_1\sin\theta\right)\big/2rR_0\Omega_s$ in the ballooning representation. For the simplicity of notation, ω_{ds} denotes the drift frequency in ballooning space. Before further analysis, it should be noted that the analysis in this work will be carried out in the inertial layer. Thus, we have two scales, i.e. $|\vartheta_1| \sim \delta^{-1} \sim \beta^{-1/2}$ and $|\vartheta_0| \sim 1$. With the separated scales, the related wave numbers can be given as $k_\vartheta = nq/r$, $k_\perp = k_\vartheta \hat{\kappa}$ and $\hat{\kappa} \simeq \sqrt{1+s^2\vartheta_1^2}$ in the inertial layer. In the original space , $\dot{\theta}$, $\dot{\zeta}$ and $\dot{r}$ are all periodic functions in θ . Since periodic functions in θ are represented as periodic function in ϑ , all the periodic functions in original space, can be treated as functions only in the short scale ϑ_0 , which indicates that the particle orbits are represented on the scale of ϑ_0 . In order to study the responses of trapped and circulating particles, the canonical angle can be introduced as[7]

$$\vartheta_c = \omega_b \int^{\vartheta_0} \frac{d\vartheta_0'}{\dot{\theta}}, \quad (7)$$

where the bouncing frequency $\omega_b = 2\pi / \left(\oint d\vartheta_0 / \dot{\theta} \right)$ .The bouncing frequency for circulating particles with $0 < \lambda < 1-\epsilon$ can be obtained as[6]

$$\omega_b = \frac{\pi v \sqrt{2\epsilon\lambda}}{2qR_0} \frac{\kappa}{K\left(\kappa^{-1}\right)},$$

where $K(\kappa)$ is the elliptic integral of the first kind, $\kappa^2 = \left[1-(1-\epsilon)\lambda\right]/2\epsilon\lambda$, and $\kappa > 1$ for circulating particles. For trapped particles with $1-\epsilon < \lambda < 1+\epsilon$, the bouncing frequency is

$$\omega_b = \frac{\pi v \sqrt{2\epsilon\lambda}}{4qR_0} K^{-1}(\kappa),$$

where $0 < \kappa < 1$ for trapped particles. The bouncing averaged drift frequency for trapped particles in intertial region can be calculated as

$$\bar{\omega}_{ds} = \frac{k_\vartheta E}{R_0 \Omega_s}(2-\lambda)\left[2\frac{E(\kappa)}{K(\kappa)} - 1\right],$$

where $E(\kappa)$ is the complete elliptic integrals of the kinds.

2.3. GOVERNING EQUATIONS

In order to obtain the dispersion relation for LFDAW, the quasi-neutrality condition and vorticity equation are adopted as governing equations. The quasi-neutrality equation can be cast as,

$$\left(\frac{e_i n_i}{T_i} + \frac{e n_i}{T_e}\right)(\delta\phi - \delta\psi) + \frac{en}{T_i}\left(1 - \frac{\omega_{*pi}}{\omega}\right) b_i \delta\psi = \left\langle \sum_{s=i,e} J_0 \delta K_s \right\rangle, \quad (8)$$

where $\langle \cdots \rangle = 2\pi \sum_\sigma \int\int B/|v_\parallel| (\cdots) dE d\mu$ is the integration of the velocity space. And the vorticity equation is obtained from the parallel Ampère's law and gyro-kinetic equation, which can be given as[4]

$$B\mathbf{b}\cdot\nabla \frac{k_\perp^2}{Bk_\vartheta^2} \mathbf{b}\cdot\nabla\delta\psi + \frac{\omega^2}{v_A^2}\left(1 - \frac{\omega_{*pi}}{\omega}\right)\frac{k_\perp^2}{k_\vartheta^2}\delta\phi + \frac{\alpha g(\vartheta)}{q^2 R_0^2}\delta\psi = \left\langle \sum_s \frac{4\pi e_s}{k_\vartheta^2 c^2} J_0 \omega \omega_{ds} \delta K_s \right\rangle, \quad (9)$$

where $v_A = B/\sqrt{4\pi n_i m_i}$ is the Alfvén velocity, $\alpha = -R_0 q^2 \beta'$, β is the ratio of plasma energy over magnetic field energy, $g(\vartheta) \simeq \cos\vartheta + s\vartheta_1 \sin\vartheta$, $\omega_{*ps} = (cT_s/e_s B)(\mathbf{k}\times\mathbf{b})\cdot\nabla n_s/n_s + (c/e_s B)(\mathbf{k}\times\mathbf{b})\cdot\nabla T_s$, T_i is the ion temperature and n_i is the ion density.

3. ANALYTICAL RESULTS

3.1. ORDERINGS

In the singular layer with $\vartheta_1 \sim \delta^{-1}$, we have the orderings $\delta\omega_{di} \sim \omega \sim \omega_{*Pi} \sim \omega_{bi}$ for circulating ions and $\omega_{de} \sim \omega_{di} \sim \omega \sim \omega_{bi} \sim \delta\omega_{be}$ for trapped ions and electrons. Moreover, the Larmor radius of ions is assumed to be small, i.e. $k_\perp \rho_{ti} \sim \delta$, in singular layer, where $\rho_{ti} = m_s c v_{ti} / e_s B_0$. The 0th order kinetic equations for circulating ions, trapped ions and trapped electrons can be written as

$$\left(\dot{\vartheta}\partial_{\vartheta_0} - i\omega\right)\delta K^{(0)}_{cir} = i\left(\frac{e}{m}QF_0\right)_i \frac{k_\vartheta}{k_\perp}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)}\right),$$

$$\left(\dot{\vartheta}\partial_{\vartheta_0} - i\omega + i\omega_{di}\right)\delta K^{(0)}_{tr,i} = i\left(\frac{e}{m}QF_0\right)_i \frac{k_\vartheta}{k_\perp}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)} + \frac{\omega_{di}}{\omega}\delta\Psi^{(0)}\right),$$

$$\left(-i\omega + i\omega_{de}\right)\delta K^{(0)}_{tr,e} = i\left(\frac{e}{m}QF_0\right)_e \frac{k_\vartheta}{k_\perp}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)} + \frac{\overline{\omega}_{de}}{\omega}\delta\Psi^{(0)}\right).$$

Here we have re-defined $\left[\delta\Psi, \delta\Phi\right] = \hat{\kappa}_\perp \left[\delta\psi, \delta\phi\right]$. The 0-th order quasi-neutrality condition and vorticity equation can be given as

$$\left(1+\frac{1}{\tau}\right)\left(\delta\Phi^{(0)} - \delta\Psi^{(0)}\right) = -\frac{T_i}{m_e N_i}\left\langle \frac{QF_{0e}}{\omega - \overline{\omega}_{de}}\left[\frac{\overline{\omega}_{de}}{\omega}\delta\Psi^{(0)} + \delta\Phi^{(0)} - \delta\Psi^{(0)}\right]\right\rangle_{tr}$$

$$-\frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)}\right)\right\rangle_{circ} - \frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega - \overline{\omega}_{di}}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)} + \frac{\overline{\omega}_{di}}{\omega}\delta\Psi^{(0)}\right)\right\rangle_{tr},$$

$$\partial^2_{\vartheta_0}\delta\Psi^{(0)} = 0, \quad (10)$$

where the average of ϑ_0 has been taken, $N_i = \left\langle F_{0i}\right\rangle_{circ} + \left\langle F_{0i}\right\rangle_{tr}$, $\left\langle \cdots \right\rangle_{tr} = \int_0^\infty \int_{1-\epsilon}^{1+\epsilon} \tau_{bs}\mathcal{E}/qR_0\left(\cdots\right) d\lambda d\mathcal{E}$ for trapped particles, $\left\langle \cdots \right\rangle_{circ} = 2\int_0^\infty \int_0^{1-\epsilon} \tau_{bs}\mathcal{E}/qR_0\left(\cdots\right) d\lambda d\mathcal{E}$ for circulating particles, $\tau = T_i/T_e$ and $\tau_{bs} = 2\pi/\omega_{bs}$. The 1st kinetic equation for circulating ions can be given as

$$\left(\omega_b\partial_{\vartheta_c} - i\omega\right)\delta K^{(1)}_{circ} = -\dot{\vartheta}\partial_{\vartheta_1}\delta K^{(0)}_{circ} + i\frac{k_\vartheta}{k_\perp}\left(\frac{e}{m}\right)QF_{0i}\left[\delta\Phi^{(1)} + \frac{\omega_{di}}{\omega}\delta\Phi^{(0)}\right].$$

By assuming *a posteriori* that $\delta\Phi \simeq \delta\Phi^{(0)} + \delta\Phi_s \sin\vartheta_0$ [3], we have the 1st order quasi-neutrality and vorticity equations are

$$\left(1+\frac{1}{\tau}\right)\delta\Phi_s = -\frac{T_i}{m_i N_i}\left\langle \frac{\left(\omega - \overline{\omega}_d\right)L^2 QF_{0i}}{\left(\omega - \overline{\omega}_{di}\right)^2 - \omega_{bi}^2}\left[\delta\Phi_s + \frac{k_\vartheta \Omega_d}{\omega}\frac{\mathcal{E}}{\mathcal{E}_t}\left(2-\lambda\right)s\vartheta_1\delta\Psi^{(0)}\right]\right\rangle_{tr}$$

$$-\frac{T_i}{m_i N_i}\left\langle \frac{M^2 QF_{0i}}{\omega^2 - \omega_{bi}^2}\left[\omega\delta\Phi_s + k_\vartheta \Omega_d \frac{\mathcal{E}}{\mathcal{E}_t}\left(2-\lambda\right)s\vartheta_1\delta\Psi^{(0)}\right]\right\rangle_{circ},$$

$$\frac{\partial^2}{\partial\vartheta_0^2}\delta\Psi^{(1)} = 0, \quad (11)$$

where $\mathcal{E}_t$ is the thermal energy of ions per mass, $L\left(\epsilon,\lambda\right) = \pi^{-1}\oint \sin\vartheta_0 \sin\vartheta_c d\vartheta_c$ for trapped ions and $M\left(\epsilon,\lambda,\sigma\right) = \pi^{-1}\oint \sin\vartheta_0 \sin\vartheta_c d\vartheta_c$ for passing ions. In general, $\sin\vartheta_0$ can couple with the harmonics of ϑ_c to any order. However, the analysis in Subsection 3.2 below can rule out most of the harmonic couplings. And the 1st order vector potential is incorporated into $\delta\Psi^{(0)}$ and hence can be taken as zero. Provided that the governing equations are solved order by order, the 2nd order vorticity equation can give the GFLDR, which is

$$\frac{\partial^2}{\partial\vartheta_1^2}\delta\Psi^{(0)} + \frac{\omega^2}{\omega_A^2}\left(1 - \frac{\omega_{*p}}{\omega}\right)\delta\Psi^{(0)} = \frac{4\pi\omega}{k_\perp k_\vartheta c^2}q^2 R_0^2\left\langle e_i\omega_{di}\delta K_i^{(1)}\right\rangle_{circ}. \quad (12)$$

Now we have all the equations need to be solved. Before conducting the analytical calculation, some useful symmetry analysis should be carried out.

3.2. UP-DOWN SYMMETRY AND PSEUDO-ORTHOGONAL RELATIONS

Since the magnetic configuration is up-down symmetric ($\theta \to -\theta$), the canonical angle can be defined piece-wisely. First, we consider the circulating particles with $\sigma = 1$. For $0 \le \vartheta_0 < \pi$,

$$\vartheta_c\left(\vartheta_0, \sigma = 1\right) = \vartheta^+_{circ}\left(\vartheta_0\right) = 2\pi\frac{\int_0^{\vartheta_0} d\vartheta_0' / \dot{\theta}}{\oint d\vartheta_0 / \dot{\theta}} = \pi\frac{K\left(\theta/2, \kappa^{-1}\right)}{K\left(\kappa^{-1}\right)}, \quad (13)$$

where $K(x,\kappa)$ is the incomplete elliptic integral for the first kind. And for $\pi \le \vartheta_0 < 2\pi$,

$$\vartheta_c(\vartheta_0,\sigma=1) = 2\pi - \vartheta_{circ}^{+}(2\pi-\vartheta_0).$$

For $\sigma=-1$, since the orbit width is neglected, we have the reflection formula

$$\vartheta_c(\vartheta_0,\sigma=-1) = 2\pi - \vartheta_c(\vartheta_0,\sigma=1).$$

For trapped particles, the canonical angle is obtained piece wisely as

$$\theta_c(\vartheta_0) = \theta_{tr}^{+}(\vartheta_0) = 2\pi \frac{\int_0^{\vartheta_0} d\vartheta_0' / \dot{\theta}}{\oint d\vartheta_0 / \dot{\theta}} = \frac{\pi K(x,\kappa)}{2K(\kappa)}, \quad \text{for} \quad 0 \le \theta_0 \le \theta_b \quad \text{and} \quad \dot{\theta} > 0, \quad (14)$$

where $\sin x = \sin(\theta/2)/\sin(\theta_b/2)$. With the up-down symmetry, the definition of canonical angle along the rest parts of the orbit can be obtained as

$$\theta_c(\vartheta_0) = \begin{cases} \pi - \theta_{tr}^{+}(\theta), & \text{for} \quad \theta_b > \theta_0 \ge 0, \dot{\theta} < 0, \\ \pi + \theta_{tr}^{+}(2\pi-\theta), & \text{for} \quad 2\pi > \theta_0 \ge 2\pi - \theta_b, \dot{\theta} < 0, \\ 2\pi - \theta_{tr}^{+}(2\pi-\theta), & \text{for} \quad 2\pi - \theta_b < \theta_0 < 2\pi, \dot{\theta} > 0. \end{cases}$$

With the definitions for ϑ_c and up-down symmetry relations above, by integrating along the orbit piece-wisely, the pseudo-orthogonality relation in bouncing average for both trapped and circulating particles can be given as

$$\oint \sin m\vartheta_0 \cos l\vartheta_c d\vartheta_c = \oint \cos m\vartheta_0 \sin l\vartheta_c d\vartheta_c = 0, \quad (15)$$

where m and l are integers. Moreover, for trapped particles and even l, we have

$$\oint \sin m\vartheta_0 \sin l\vartheta_c d\vartheta_c = \oint \cos m\vartheta_0 \cos\left[(l+1)\vartheta_c\right] d\vartheta_0 = 0. \quad (16)$$

As shown above, the Fourier components in ϑ_0 and ϑ_c, by averaging over ϑ_c, have the orthogonality similar to the Fourier components only in ϑ_0. Thus, the higher harmonic couplings of ϑ_c with $\sin\vartheta_0$ can be ruled out by pseudo-orthogonality together with the numerical results in Subsection 4.1.

3.3. ANALYTICAL SOLUTION

The 0th order solution of the gyro-kinetic equation can be given as

$$\delta K_{cir}^{(0)} = -\left(\frac{e}{m}\right)_i \frac{QF_{0i}}{\omega} \frac{k_\vartheta}{k_\perp}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)}\right),$$

$$\delta K_{tr,e}^{(0)} = -QF_{0e}\left(\frac{e}{m}\right)_e \frac{k_\vartheta / k_\perp}{\omega - \bar{\omega}_{de}}\left[\frac{\bar{\omega}_{de}}{\omega}\delta\Psi^{(0)} + \delta\Phi^{(0)} - \delta\Psi^{(0)}\right],$$

$$\delta K_{tr,i}^{(0)} = [\cdots]\cos\vartheta_c - \frac{1}{\omega - \bar{\omega}_{di}} \frac{e}{m} QF_{0i} \frac{k_\vartheta}{k_\perp}\left(\delta\Phi^{(0)} - \delta\Psi^{(0)} + \frac{\bar{\omega}_{di}}{\omega}\delta\Psi^{(0)}\right)$$

$$-\frac{e}{m} QF_{0i} L \frac{(\omega - \bar{\omega}_{di}) k_\vartheta / k_\perp}{(\omega - \bar{\omega}_{di})^2 - \omega_{bi}^2}\left[\delta\Phi_s + \frac{k_\vartheta \Omega_d}{\omega}\frac{\mathcal{E}}{\mathcal{E}_t}(2-\lambda)s\vartheta_1\delta\Psi^{(0)}\right]\sin\vartheta_c$$

where the terms proportional to $\cos\vartheta_c$ is irrelevant due to the pseudo-orthogonal relations. The 0th order quasi-neutrality equation gives

$$\delta\Phi^{(0)} = \frac{D_1}{D_2}\delta\Psi^{(0)},$$

where

$$D_1 = \left(1+\frac{1}{\tau}\right) - \frac{T_i}{m_e N_i}\left\langle \frac{QF_{0e}}{\omega - \bar{\omega}_{de}}\left(\frac{\bar{\omega}_{de}}{\omega} - 1\right)\right\rangle_{tr} + \frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega}\right\rangle_{circ} - \frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega - \bar{\omega}_{di}}\left(\frac{\bar{\omega}_{di}}{\omega} - 1\right)\right\rangle_{tr},$$

$$D_2 = \left(1+\frac{1}{\tau}\right) + \frac{T_i}{m_e N_i}\left\langle \frac{QF_{0e}}{\omega - \bar{\omega}_{de}}\right\rangle_{tr} + \frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega}\right\rangle_{circ} + \frac{T_i}{m_i N_i}\left\langle \frac{QF_{0i}}{\omega - \bar{\omega}_{di}}\right\rangle_{tr}.$$

And the solution to the1st order gyro-kinetic equation is

$$\delta K_{circ}^{(1)} = -\left(\frac{e}{m}\right)\frac{1}{\omega^2 - \omega_{bi}^2}\frac{k_\vartheta}{k_\perp} QF_{0i} M\left[\omega\delta\Phi_s + \frac{\mathcal{E}}{\mathcal{E}_t}k_\vartheta\Omega_d(2-\lambda)s\vartheta_1\delta\Psi^{(0)}\right]\sin\vartheta_c + [\cdots]\cos\vartheta_c.$$

And the 1st quasi-neutrality equation gives

$$\delta\Phi_s = N_s \delta\Psi^{(0)} = \frac{k_\vartheta s \vartheta_1 \Omega_d D_3}{\omega D_4} \delta\Psi^{(0)},$$

where

$$D_3 = -\frac{T_i}{m_i N_i}\left\langle \frac{(\omega-\bar{\omega}_{di}) L^2 Q F_{0i}}{(\omega-\bar{\omega}_{di})^2 - \omega_{bi}^2} \frac{\mathcal{E}}{\mathcal{E}_t}(2-\lambda)\right\rangle_{tr} - \frac{T_i}{m_i N_i}\left\langle \frac{M^2 \omega Q F_{0i}}{\omega^2 - \omega_{bi}^2} \frac{\mathcal{E}}{\mathcal{E}_t}(2-\lambda)\right\rangle_{circ},$$

$$D_4 = \left(1+\frac{1}{\tau}\right) + \frac{T_i}{m_i N_i}\left\langle \frac{M^2 \omega Q F_{0i}}{\omega^2 - \omega_{bi}^2}\right\rangle_{circ} + \frac{T_i}{m_i N_i}\left\langle \frac{(\omega-\bar{\omega}_{di}) L^2 Q F_{0i}}{(\omega-\bar{\omega}_{di})^2 - \omega_{bi}^2}\right\rangle_{tr}.$$

Finally, the second order vorticity equation is obtained as

$$\frac{\partial^2 \delta\Psi^{(0)}}{\partial \vartheta_1^2} + \Lambda^2 \delta\Psi^{(0)} = 0, \quad (17)$$

where $\Lambda^2 = \Lambda_{fl}^2 + \Lambda_{circ}^2 + \Lambda_{tr}^2$, $I(E,\lambda) = \mathcal{E}/\mathcal{E}_t(2-\lambda)$,

$$\Lambda_{fl}^2 = -\frac{8\pi^2\omega}{B_0^2} q^2 R_0^2 N_0 \sum_s \left\langle m_s Q F_{0s} \mathcal{E} \lambda^2 \right\rangle,$$

$$\Lambda_{circ}^2 = \frac{2\pi\omega e^2}{m_i c^2} q^2 R_0^2 \left\langle \frac{Q F_{0i} M^2 \Omega_d^2}{\omega^2 - \omega_{bi}^2}\left(N_s I + I^2\right)\right\rangle_{circ},$$

and

$$\Lambda_{tr}^2 = \frac{2\pi\omega e^2}{m_i c^2} q^2 R_0^2 \left\langle \frac{(\omega-\bar{\omega}_{di}) Q F_{0i} L^2}{(\omega-\bar{\omega}_{di})^2 - \omega_{bi}^2} \frac{\Omega_d^2}{\omega}\left(N_s I + I^2\right)\right\rangle_{tr}.$$

Now the inertia term Λ^2 contains the corrections from the contribution of particles near passing trapped separatrix. As shown in the results above, the particles near the barely passing trapped separatrix can modify the inertia term. By taking the equilibrium distribution function as Maxwellian, the generalized inertia can be represented by the plasma dispersion relations.

4. LIMITING CASES

4.1. DEEP TRAP LIMIT

In this subsection, we will analyze the weight functions, i.e. M and L to demonstrate how the results are related to those in previous researches[3]. If the circulating particles are approximately taken to be well circulating, i.e. $\lambda \ll 1$, the canonical angle is then

$$\vartheta_c(\vartheta_0, \sigma = \pm 1) = 2\pi\delta_{-1,\sigma} + \sigma\vartheta_0. \quad (18)$$

And the parallel velocity $v_\parallel$ can be treated approximately independent of ϑ_0. Then the weight term M can be given as

$$M(\sigma = 1) \simeq \frac{1}{\pi}\int_0^{2\pi} \sin\vartheta_0 \sin\vartheta_0 d\vartheta_0 = 1. (19)$$

And the velocity space integral is

$$\langle \cdots \rangle_{circ} \simeq 4\pi \int_0^\infty \int_0^{1-\epsilon} (\cdots) \sqrt{\frac{\mathcal{E}}{2(1-\lambda)}} d\lambda d\mathcal{E}. \quad (20)$$

For the trapped particles, by assuming trapped particles are deeply trapped, we have $\theta_b \simeq 2\kappa$, $\vartheta_0 \simeq \theta_b \sin\vartheta_c$, $v_\parallel \simeq \theta_b \omega_b q R_0 \cos\vartheta_c$ and $\omega_b = \sqrt{\epsilon\mu B_0}/qR_0$. Then the weight term L and the velocity space integral $\langle \cdots \rangle_{tr}$ can be given as

$$L = \frac{4}{\pi}\int_0^{\pi/2} \sin\vartheta_c \sin\vartheta_0 d\vartheta_c \simeq \theta_b, (21)$$

and

$$\langle \cdots \rangle_{tr} \simeq \sqrt{\epsilon}4\pi\int_0^\infty [\cdots]\mathcal{E}^{1/2}d\mathcal{E}. (22)$$

Here the approximation $\sin\vartheta_0 \simeq \theta_b \sin\vartheta_c$ has been used. As shown in Fig. 1, $M \simeq 1$ except for the circulating ions near circulating/trapped separatrix, which is illustrated by comparing with the higher coupling weight function $M' = \pi^{-1}\oint \sin\vartheta_0 \sin 2\vartheta_c d\vartheta_c$. As shown in Fig. 1, $\vartheta_c \simeq \vartheta_0$ is a good approximation for the majority of circulating ions. And the demonstration that the higher harmonic of ϑ_c can be ruled out is now complete. In Fig.1, it is illustrated that the value of L/θ_b approaches 1 for deeply trapped particles. Moreover, the value of the L function peaks at the medium trapped region, which indicates that the contribution from the medium trapped particles is bigger than that of the deeply trapped particles. Finally, with the analysis above, the results in Subsection 3.3 directly go back the previous results with well-circulating and deeply trapped particles[3]. As shown in Fig. 1, the weight functions L and M of the particles near passing/trapped boundary and deeply trapped region is small. Thus, the well passing and medium trapped particles contribute most to the neoclassical corrections.

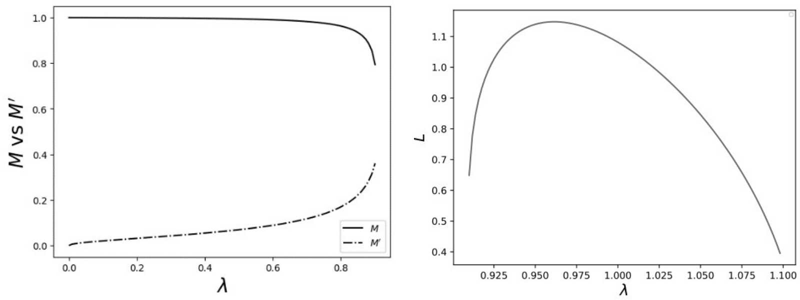

Figure 1. Value of weight function versus pitch angle variable λ for ϵ=0.1.

4.2. FLUID LIMIT

By taking the fluid limit $\omega \gg \omega_{*ps} \sim \bar{\omega}_{di} \sim \omega_{bi}$, we have $\delta\Phi^{(0)} \simeq \delta\Psi^{(0)}$ and

$$D_3 \simeq \frac{3\sqrt{2}n_i}{4\pi N_i}\int_{1-\epsilon}^{1+\epsilon}\frac{L^2K(\kappa)}{\sqrt{\epsilon\lambda}}(2-\lambda)d\lambda + \frac{3\sqrt{2}n_i}{4\pi N_i}\int_0^{1-\epsilon}\frac{M^2K(\kappa^{-1})}{\kappa\sqrt{\epsilon\lambda}}(2-\lambda)d\lambda,$$

$$D_4 \simeq \left(1+\frac{1}{\tau}\right) - \frac{\sqrt{2}n_i}{\pi N_i}\int_{1-\epsilon}^{1+\epsilon}\frac{L^2K(\kappa)}{\sqrt{\epsilon\lambda}}\frac{\Gamma(3/2)}{\sqrt{\pi}}d\lambda - \frac{\sqrt{2}n_i}{\pi N_i}\int_0^{1-\epsilon}\frac{M^2K(\kappa^{-1})}{\kappa\sqrt{\epsilon\lambda}}\frac{\Gamma(3/2)}{\sqrt{\pi}}d\lambda,$$

and

$$\frac{n_i}{N_i} = \frac{n_e}{N_i} = \sqrt{2}\pi\left(\int_0^{1-\epsilon}\frac{K(\kappa^{-1})}{\kappa\sqrt{\epsilon\lambda}}d\lambda + \int_{1-\epsilon}^{1+\epsilon}\frac{K(\kappa)}{\sqrt{\epsilon\lambda}}d\lambda\right)^{-1}.$$

The generalized inertia can be given as

$$\begin{aligned}\Lambda^2 \simeq\ & \frac{\omega(\omega-\omega_{*Pi})}{\omega_A^2}\left(\int_0^{1-\epsilon}\lambda^{1/2}\frac{3\sqrt{2}K(\kappa^{-1})}{2\kappa\sqrt{\epsilon}}d\lambda + \int_{1-\epsilon}^{1+\epsilon}\frac{3\sqrt{2}\lambda^{1/2}K(\kappa)}{2\sqrt{\epsilon}}d\lambda\right)\\ & - \frac{\sqrt{2}\omega_{ti}^2}{4\pi\omega_A^2}q^2\int_0^{1-\epsilon}\frac{K(\kappa^{-1})M^2}{\kappa\sqrt{\epsilon\lambda}}(2-\lambda)\left[\frac{3}{4}N_s + (2-\lambda)\frac{15}{8}\right]d\lambda \\ & - \frac{\sqrt{2}\omega_{ti}^2}{4\pi\omega_A^2}q^2\int_{1-\epsilon}^{1+\epsilon}\frac{K(\kappa)L^2}{\sqrt{\epsilon\lambda}}(2-\lambda)\left[\frac{3}{4}N_s + \frac{15}{8}(2-\lambda)\right]d\lambda,\end{aligned} \qquad (23)$$

where $\omega_{ti} = v_{ti}/qR_0$ and $\omega_A = v_A/qR_0$. By taking $\Lambda^2 = 0$, the BAE frequency can be calculated numerically as in Fig. 2. And an interpolate formula can be given as

$$\omega_{BAE} \simeq q\omega_{ti}\sqrt{\tau+\frac{7}{4}-1.7\tau\sqrt{\epsilon}-0.2\sqrt{\epsilon}}. \quad (24)$$

As shown in Fig. 2, the neoclassical effects can lower the frequency of BAE. And this correction is absent if we only consider the deeply trapped and well passing particles. In experiments, the frequency of BAE is detected along radial coordinate. With the Eq. (24), the modified BAE frequency in our work can be easily verified by comparing the spectrum data from experiments.

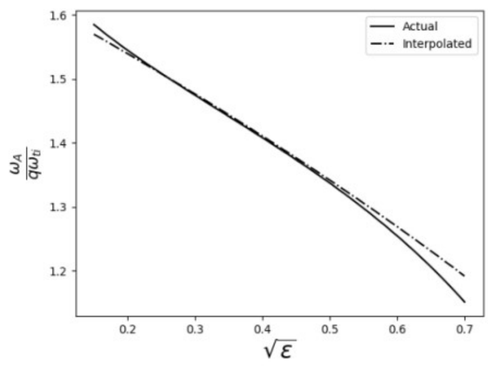

Figure 2. Neoclassical correction on the frequency of BAE for τ≈1.

5. SUMMARY

In this work, a kinetic model with full neoclassical effects is developed. And novel inspections on the symmetry of particle orbits are introduced to make the analytical calculation accessible. The results can go back to those in the previous researches by taking the corresponding limit. Moreover, it is shown that the neoclassical effects can lower the frequency of BAE, which is absent in the deep trap limit calculation in the previous research.

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