Revealing stickiness in a field line map using recurrence analysis

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Introduction

In the pursuit of controlled thermonuclear fusion, the Hamiltonian description can be employed to study magnetic field lines in Tokamaks. In the context of symmetric magnetohydrodynamics equilibrium, the magnetic field lines are governed by the Hamiltonian canonical equations, where the ignorable coordinate assumes the role of time. Consequently, these intricate systems can be effectively described using symplectic two-dimensional nonlinear maps. Therefore, the magnetic fields exibiths the complex dynamics, showing for example the stickiness effect that causes a change in the escape times of trajectories, impacting the effective transport fluxes.

In this work, we used RPs to characterize the dynamics of a nonlinear system, applying RQA to identify regular, chaotic, and sticky regions in a magnetic field lines model called Tokamap [1]. The RQA indicator we choose is based on the Shannon entropy of recurrence times, known as the recurrence time entropy (RTE) [2].

Tokamap and recurrence plots

A paradigmatic field line map was proposed by Balescu and coworkers, the *Tokamap* [1], given by the equations

$$\psi_{n+1} = \frac{1}{2}\Big\{P(\psi_n,\theta_n) + \sqrt{[P(\psi_n,\theta_n)]^2 + 4\psi_n},\Big\}, \quad (1)$$

$$P(\psi_n,\theta_n) = \psi_n - 1 - \frac{k}{2\pi}\sin(2\pi\theta_n), \quad (2)$$

$$\theta_{n+1} = \theta_n + \frac{1}{q(\psi_n)} - \frac{k}{4\pi^2}\frac{1}{(1+\psi_{n+1})^2}\cos(2\pi\theta_n) \qquad (\mathrm{mod}\,1) \quad (3)$$

$$q(\psi) = \frac{4}{(2-\psi)(2-2\psi+\psi^2)}, \quad (4)$$

where $k > 0$ is a variable parameter which stands for the strength of the non-symmetrical perturbation acting upon the Tokamak.

Some chaotic orbits spend a considerable amount of time around periodic islands, exhibiting the stickiness effect. Although a sticky orbit is also chaotic and area-filling, it occupies a slimmer region of the phase space. How to distinguish between a sticky and a non-sticky chaotic orbit is the main goal of the present work.

Given a time series corresponding to a map orbit $\mathbf{x}_i = (\psi_i, \theta_j)^T$ $(i = 1, 2, \ldots, N)$, the corresponding recurrence matrix elements are defined as $R_{ij} = \Theta(\varepsilon - \|\mathbf{x}_i - \mathbf{x}_j\|)$, where N is the length of the time series, ε is the threshold parameter, Θ is the Heaviside function, and $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the spatial distance between two states at different times, here calculated with the supremum (or maximum) norm.

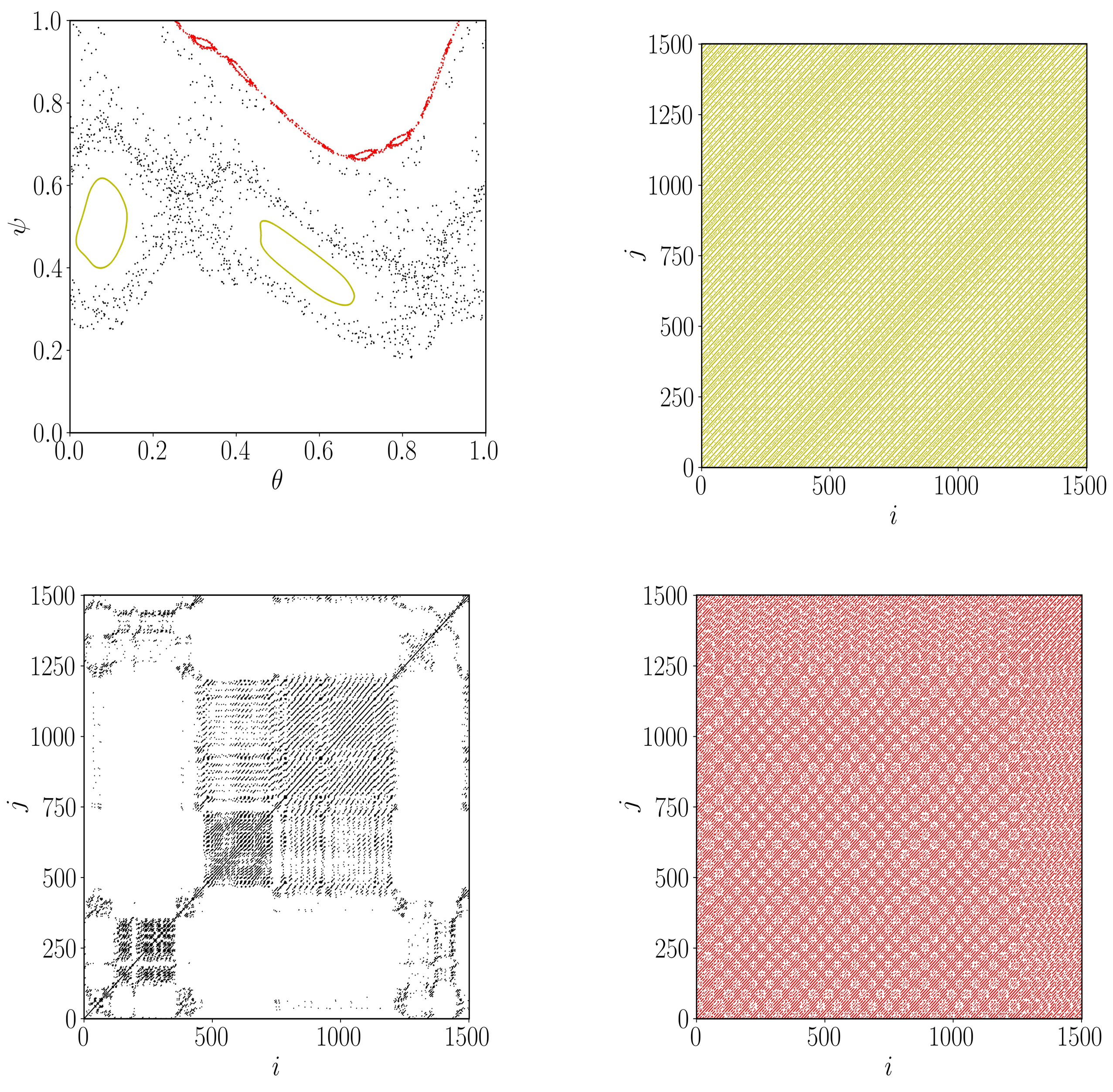

Figure 1: (a) Phase space with a quasiperiodic orbit (yellow), a chaotic orbit (black), and a sticky orbit (red) of the Tokamap with $k = \frac{3\pi}{2}$ iterated for $N = 1500$ times. Recurrence matrix of the (b) quasiperiodic orbit, (c) the chaotic orbit, and (d) the sticky orbit.

We define the Recurrence Time Entropy RTE,

$$RTE = -\sum_{\nu=\nu_{min}}^{\nu_{max}} p_w(\nu)\ln p_w(\nu). \quad (5)$$

The RTE is able to characterize the dynamics as a consequence of Slater's theorem. By counting the number of return times (or recurrence times) or in this case the white vertical lines, we can distinguish between the different kinds of solutions of a nonlinear system. If it is one, the orbit is periodic, what correspond to RTE = 0, and if it is equal to three, the orbit is quasiperiodic, resulting in a small RTE. If the number of return times is larger than three, then the orbit is chaotic, corresponding to a large RTE. The RTE of sticky orbits is smaller than the chaotic ones, but higher than a quasiperiodic orbit.

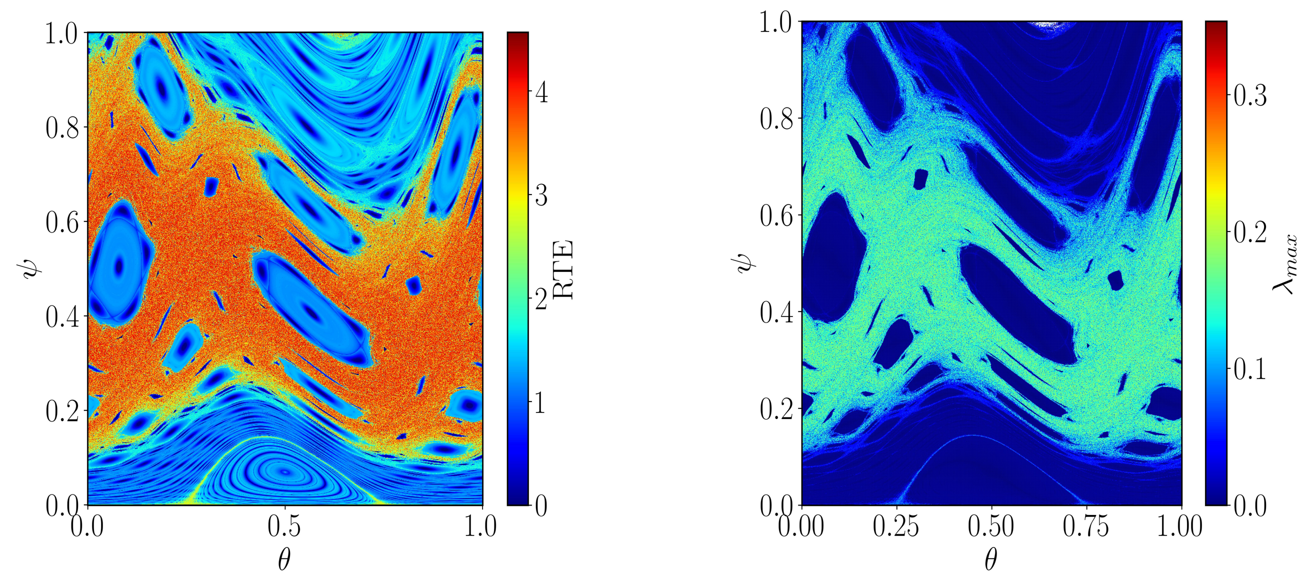

Figure 2: (a) Recurrence time entropy and (b) largest Lyapunov exponent, for the Tokamap with $k = 4.5$ iterated for $N = 10^3$ times.

The RTE and the Lyapunov exponents are positively correlated. In the chaotic sea, λ_{max} is large and so is the RTE. In the islands, λ_{max} goes to zero and the RTE is low. Moreover, in the regions, where the rotation number of the orbit is close to a rational number the RTE is smaller, dark blue. Furthermore, in the neighborhood of the islands the RTE takes intermediary values, indicating that in these regions the orbits get trapped.

Stickiness, Recurrence Time Entropy, and Trapping Time Distribution

We calculated the RTE for a single chaotic orbit in windows of size n, RTE$(i)(n)$ $i = 1, 2, \ldots, M$, where $M = N/n$, and define the probability distribution of the finite-time RTE, $P(\mathrm{RTE}(n))$, by computing a frequency histogram of RTE$(i)(n)$ such as for $N = 10^{11}$ and $n = 200$. The changes in the RTE(200) values indicates changes of dynamical behavior, which are the causes of the many modes observed in the corresponding probability distribution.

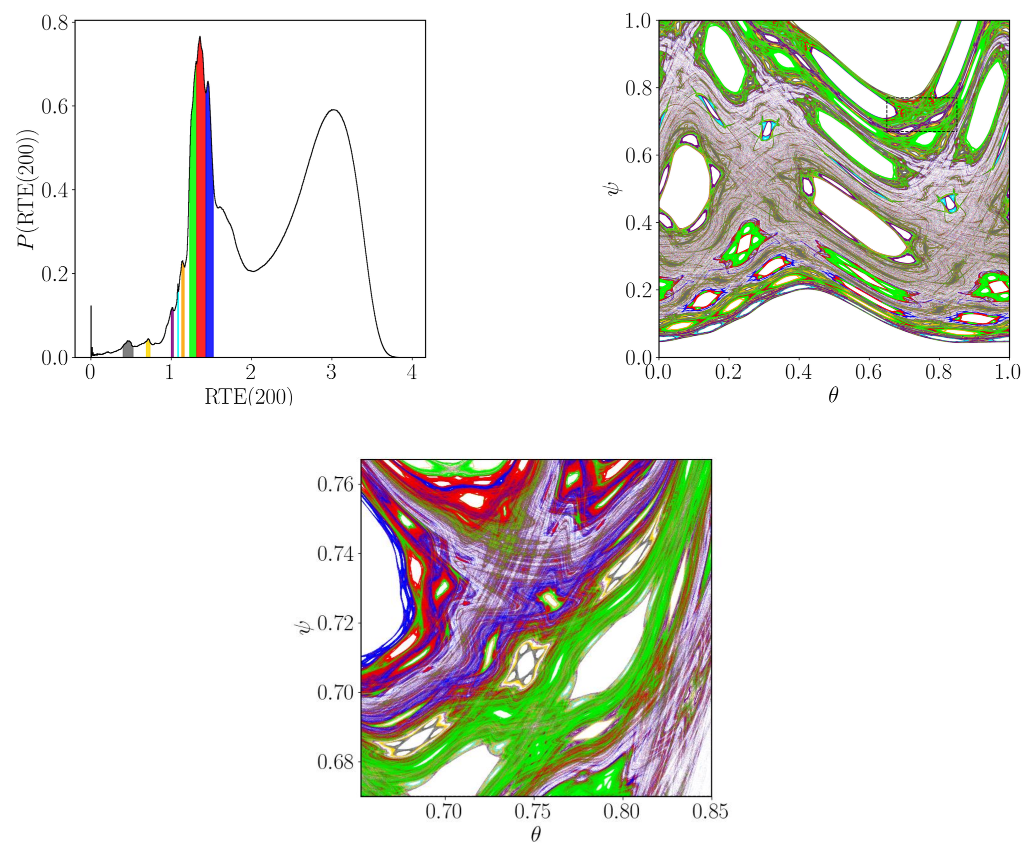

Figure 3: (a) (a) Finite-time RTE distribution for a single chaotic orbit, with $n = 200$, $N = 10^{11}$, and $k = 3\pi/2$. (b) the phase space points that generate the small RTE(200) values peaks in (a). (c) is a magnification of the region indicated by the black dashed lines. (d) The phase space that generate the larger peak for high values of RTE in the distribution.

Conclusions

- In a confinement device, the magnetic field lines may describe chaotic motion, which display the stickiness effect, affecting the transport;
- It is possible to distinguish between regular and chaotic motion using RTE;
- The RTE is positively correlated with the maximum Lyapunov exponent. Moreover, RTE is able to capture more details of the phase space;
- That is, a multi-modal distribution of RTE, where each peak correspond to a different level of a hierarchical structure of stickiness.

Acknowledgements

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References

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[2] M. R. Sales, M. Mugnaine, J. D. Szezech Jr, R. L. Viana, I. L. Caldas, N. Marwan, and J. Kurths. Stickiness and recurrence plots: an entropy-based approach. *Chaos*, 33:033140, 2023.