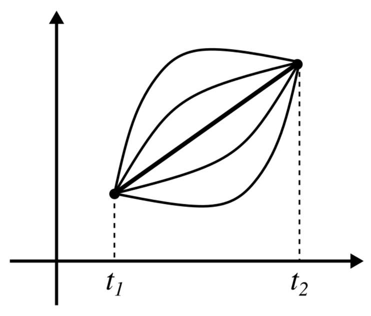
Description of magnetic field lines in fusion plasmas by the Hamiltonian formalism

Michele Mugnaine[1], Ricardo L. Viana[2], Iberê L. Caldas[1]
[1]Institute of Physics, University of São Paulo, São Paulo - SP, Brazil
[2] Department of Physics, Federal University of Paraná, Curitiba - BR, Brazil
mmugnaine@gmail.com

Variational Principle

$$\delta \int_{t_1}^{t_2} L(q,\dot q,t)dt = 0$$



Lagrangian

$$L = \frac{1}{2}mv^2 - e\Phi + e\mathbf{A}\cdot\mathbf{v}$$

$$\xrightarrow{m=0,\Phi=0}$$

$$\delta \int_{t_1}^{t_2} \mathbf{A}\cdot\mathbf{v}\,dt = 0 \rightarrow \delta \int_{t_1}^{t_2} \mathbf{A}\cdot d\mathbf{r} = 0$$

Gauge $A_2 = 0$

$$\delta \int_1^2 (A_1 dx^1 + A_3 dx^3) = 0$$

Hamiltonian description

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \xrightarrow{H=\mathbf{p}\cdot\dot{\mathbf{q}}-L(\mathbf{q},\dot{\mathbf{q}},t)} \delta \int_{t_1}^{t_2}\{p\dot q - H\}dt = 0$$

$$\delta \int_1^2 \{p\,dq - H(p,q,t)\,dt\} = 0 \qquad q = x^1,\; p = A_1(x^1,x^2,x^3) \qquad t = x^3,\; H = -A_3(x^1,x^2,x^3)$$

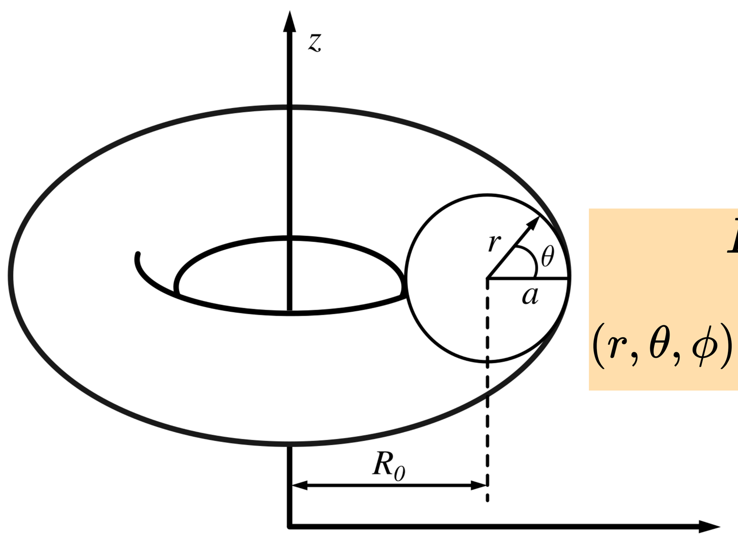
$$B = \nabla\times\mathbf{A}$$

$$B^1 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) \quad B^2 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right) \quad B^3 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right)$$

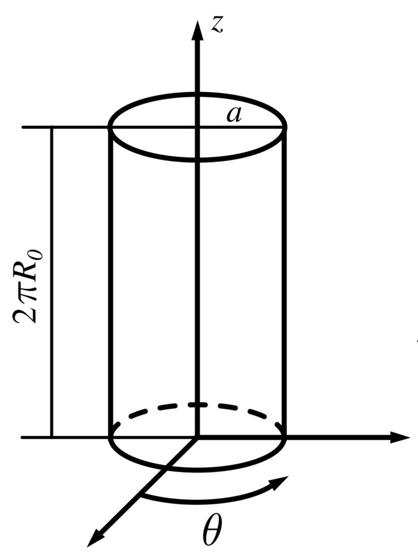
$$A_1 = -\int \sqrt{g}\,B^3 dx^2 \qquad A_3 = \int \sqrt{g}\,B^1 dx^2$$

$$H = -\int \sqrt{g}\,B^1 dx^2$$

Cylindrical coordinates



$$R_0 \gg a \qquad (r,\theta,\phi) \xrightarrow{z=R_0\phi} (r,\theta,z)$$



$$x^1 = \theta,\quad x^2 = r,\quad x^3 = z$$

$$B_{\langle 1\rangle} = rB^1 = B_\theta,\quad B_{\langle 2\rangle} = B^2 = B_r,\quad B_{\langle 3\rangle} = B^3 = B_z$$

$$g_{11} = r^2,\; g_{22} = g_{33} = 1 \qquad B_{\langle i\rangle} = \sqrt{g_{ii}}\,B^i$$

$$q = \theta \qquad t = z = x^3 \qquad p = -\int B_z r\,dr \qquad H = -\int B_\theta dr$$

$$B_r = 0,\; B_\theta = B_\theta(r),\; B_z = B_0$$

$$\mathbf{B}\times d\mathbf{r} = \mathbf{0} \rightarrow \frac{dx^1}{B^1} = \frac{dx^2}{B^2} = \frac{dx^3}{B^3} \rightarrow \frac{d\theta}{B_\theta/r} = \frac{dz}{B_z} = \frac{R_0 d\phi}{B_0}$$

$$q_{sf}(r) = \frac{d\phi}{d\theta} = \frac{rB_0}{R_0B_\theta(r)} \rightarrow B_\theta(r) = \frac{rB_0}{R_0 q_{sf}(r)}$$

$$H = -\frac{B_0}{R_0}\int \frac{r dr}{q_{sf}(r)} \xrightarrow{p=-B_0\int r dr = \frac{1}{2}B_0 r^2} H = \frac{1}{R_0}\int \frac{dp}{q(p)}$$

$$(p,q,t) \rightarrow \left(J = -\frac{p}{B_0} = \frac{r^2}{2}, \theta, \phi = \frac{t}{R_0}\right) \qquad H_0 = -\frac{R_0 H}{B_0}$$

$$H_0 = \int \frac{dJ}{q_{sf}(J)}$$

Example 1

- Plasma column of radius a, for which the electric current density is

$$j_z(r) = j_0\left(1 - \frac{r^2}{a^2}\right), \qquad j_0 = \frac{2I_p}{\pi a^2}$$

$$B_\theta(r) = B_{\theta a}\frac{r}{a}\left(2 - \frac{r^2}{a^2}\right), \qquad B_{\theta a} = \frac{\mu_0 I_p}{2\pi a}$$

$$q_{sf}(r) = q_a\left(2 - \frac{r^2}{a^2}\right)^{-1}, \qquad q_a = \frac{2\pi a^2 B_0}{\mu_0 R_0 I_p}$$

$$J = \frac{r^2}{2},\; H = \int \frac{dJ}{q_{sf}(J)}$$

$$H_0(J) = \frac{2J}{q_a}\left(1 - \frac{J}{2a^2}\right)$$

$$\frac{d\theta}{d\phi} = \frac{\partial H_0}{\partial J}; \qquad \frac{dJ}{d\phi} = -\frac{\partial H_0}{\partial \theta} = 0$$

Example 2

- Large aspect-ratio tokamak with an ergodic magnetic limiter

$$B_r^{(1)}(r,\theta,\phi) = -\frac{\mu_0 m I_L}{\pi a^m} r^{m-1}\sin(m\theta)f(\phi)$$

$$B_\theta^{(1)}(r,\theta,\phi) = -\frac{\mu_0 m I_L}{\pi a^m} r^{m-1}\cos(m\theta)f(\phi)$$

$$f(\phi) = \begin{cases} 1, \text{if } 0 \le \phi < \ell/R_0, \\ 0, \text{if } \ell/R_0 < \phi < 2\pi. \end{cases}$$

$$J = \frac{r^2}{2}, \quad q_{sf} = \frac{rB_0}{R_0 B_\theta(r)}, \quad H = \int \frac{dJ}{q_{sf}(J)}$$

$$H = H_0 + H_1$$

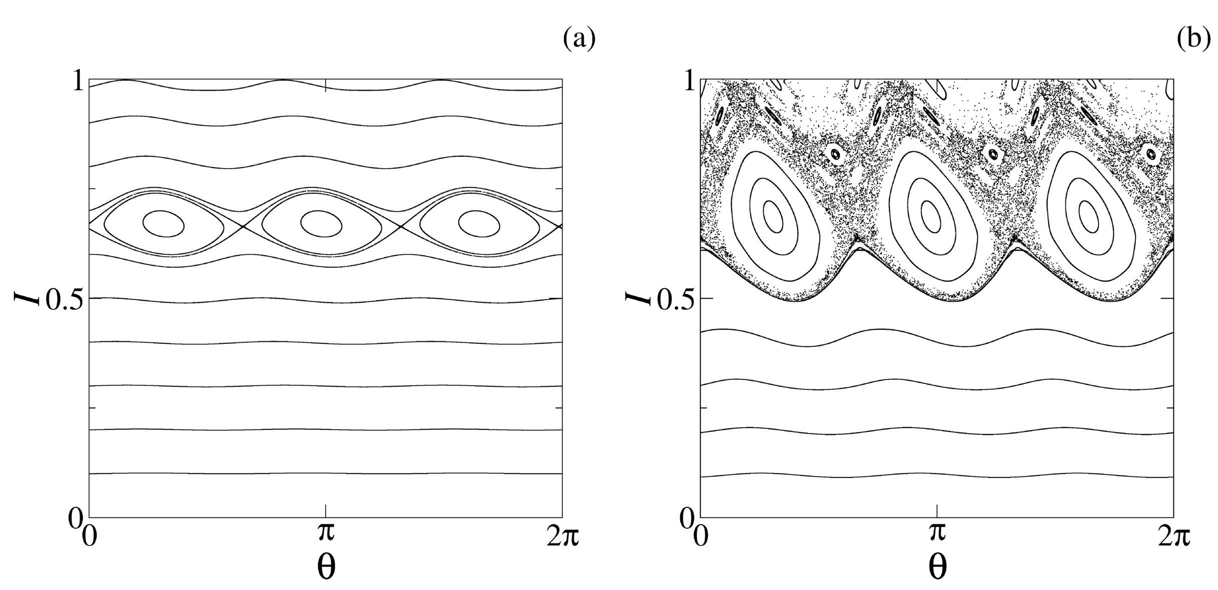
$$H = \frac{2J}{q_a}\left(1 - \frac{J}{2a^2}\right) - \frac{\mu_0 R_0 I_L}{B_0 \pi a^m}(2J)^{\frac{m}{2}}\cos(m\theta)f(\phi)$$

$$\varepsilon = \frac{I_L}{I_p},\; I = \frac{J}{a^2/2},\; \mathcal{H} = \frac{H}{a^2/2}$$

$$\mathcal{H}(I,\theta,\phi) = I\left(1 - \frac{I}{4}\right) - 2\varepsilon I^{\frac{m}{2}}\cos(m\theta)f(\phi)$$

$$\frac{d\theta}{d\phi} = \frac{\partial \mathcal{H}}{\partial I} = 1 - \frac{I}{2} - m\varepsilon I^{\frac{m}{2}-1}\cos(m\theta)f(\phi)$$

$$\frac{dI}{d\phi} = -\frac{\partial \mathcal{H}}{\partial \theta} = -2m\varepsilon I^{\frac{m}{2}}\sin(m\theta)f(\phi)$$



$m = 3, \xi = 0.163$

(a)$\varepsilon = 0.025$, (b)$\varepsilon = 0.15$

Conclusions

- The Hamiltonian description of magnetic field lines is widely used for magnetic confined plasmas, allowing the use of the powerful methods of Hamiltonian theory to interpret the results and characterize the dynamic regimes observed in experiments and computational simulations
- Magnetic field lines are a non-mechanical example of a systemthat can be described by the Hamiltonian formalism.

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