Mori-Zwanzig projection operator method as a statistical correlation analysis of time-series data

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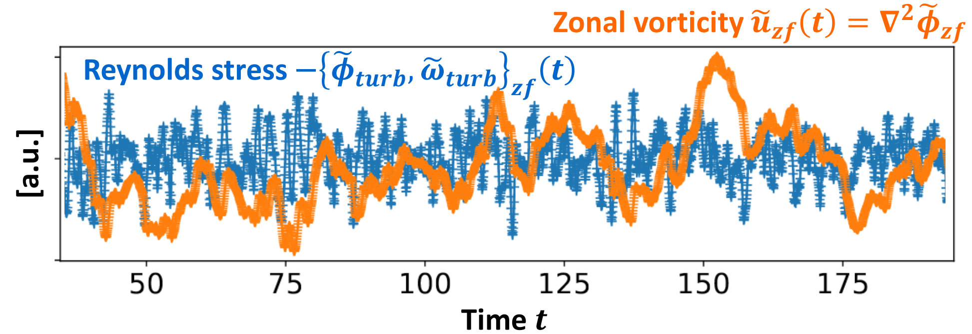
Introduction

Problem to be discussed

Q. How can we extract essential features from time-series data of zonal flows and Reynolds stress?



$$\frac{\partial \tilde{u}_{zf}}{\partial t} = -\nu k_x^2 \tilde{u}_{zf} - \{\tilde{\phi}_{turb}, \tilde{u}_{turb}\}_{zf}$$

Energy budget $\frac{\partial E_{zf}}{\partial t} = -\nu k_x^2 E_{zf} + T_{zf}$ only tells that there is time-averaged energy gain from Reynolds stress $\langle T_{zf} \rangle > 0$, balancing to dissipation $-\nu k_x^2 \langle E_{zf} \rangle < 0$ in steady state.

Q. Can we regard the Reynolds stress just as a random forcing on zonal flows?

$$\frac{\partial \tilde{u}_{zf}}{\partial t} = -\nu k_x^2 \tilde{u}_{zf} + \tilde{\xi}(t) \text{ [a random noise?]}$$

Outline

- Mori-Zwanzig projection operator method
- Application to turbulence and zonal flow interactions in drift wave turbulence
- Summary

Mori-Zwanzig projection operator method

Consider simultaneous nonlinear differential equations $\frac{d\boldsymbol{u}(t)}{dt} = \boldsymbol{N}\big(\boldsymbol{u}(t)\big)$.

A functional of $\boldsymbol{u}(t)$, i.e., $\boldsymbol{f}(t) = F\big(\boldsymbol{u}(t)\big)$ can be represented in a form of (extended) generalized Langevin equation,

$$\boldsymbol{f}(t) = \Omega \cdot \boldsymbol{u}(t) + \boldsymbol{M}(t) + \boldsymbol{r}(t), \qquad \text{with } \boldsymbol{M}(t) = -\int_0^t \Gamma(s) \cdot \boldsymbol{u}(t-s) ds$$

where, using a statistical average $\langle \cdots \rangle$,

Markov correlation: $\Omega = \langle \boldsymbol{f}\boldsymbol{u}^* \rangle \cdot \langle \boldsymbol{u}\boldsymbol{u}^* \rangle^{-1}$ Linear approx. of conditional expectation

Memory function: $\Gamma(t) = \langle \boldsymbol{r}(t) \frac{d\boldsymbol{u}^*}{dt} \rangle \cdot \langle \boldsymbol{u}\boldsymbol{u}^* \rangle^{-1}$ [Generalized 2nd FDT]

Uncorrelated term $\boldsymbol{r}(t)$: $\langle \boldsymbol{r}(t)\boldsymbol{u}^* \rangle = 0$

† This is NOT an approximation. It is a rigorous expansion [Mori'65PTP], where a time-series data $\boldsymbol{f}(t)$ is split into correlated and uncorrelated terms with respect to a time-series data $\boldsymbol{u}(t)$.

†† For numerical implementation, see [Maeyama'20JPSJ].
The function is openly available at https://github.com/smaeyama/mzprojection

Application to turbulence-zonal flow interactions

Hasegawa-Wakatani eq. [Wakatani'84PoF]

$$\frac{\partial \tilde{u}}{\partial t} + \{\tilde{\phi}, \tilde{u}\} = C(\tilde{\phi} - \tilde{n}) - \nu \nabla^4 \tilde{u}$$

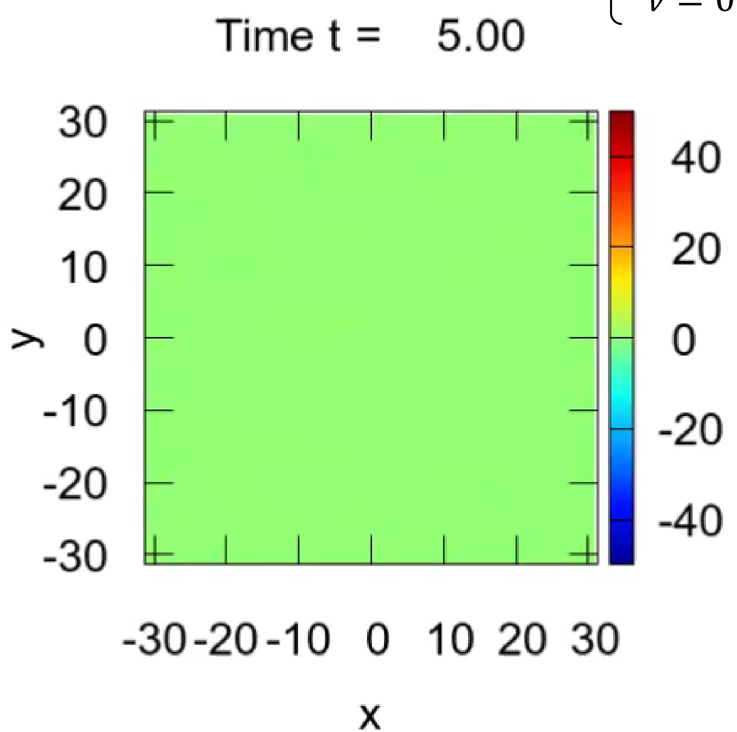
$$\frac{\partial \tilde{n}}{\partial t} + \{\tilde{\phi}, \tilde{n}\} + \kappa \frac{\partial \tilde{\phi}}{\partial y} = C(\tilde{\phi} - \tilde{n}) - \nu \nabla^4 \tilde{n}$$

with $\tilde{u} = \nabla^2 \tilde{\phi}$

- Model for electrostatic drift waves in $\boldsymbol{B}_0 = B_0(\hat{\boldsymbol{z}} + \theta \hat{\boldsymbol{y}})$, $d \ln n_0 / dx = \text{const.}$
- Describe interactions between resistive drift wave turbulence ($k_y \neq 0$) and zonal flows ($k_y = 0$)

Electrostatic potential in H-W 2D turbulence

$$\left(\begin{array}{l} C = \nabla_\parallel^2 = 3\partial_y^2, \\ \kappa = 5, \\ \nu = 0.01 \end{array} \right)$$



Evaluate zonal flows and turbulent interactions from DNS data

Taking zonal average ($k_y = 0$),

$$\frac{\partial \tilde{u}^{(ZF)}}{\partial t} + \nu \nabla^4 \tilde{u}^{(ZF)} = -\left\{\tilde{\phi}^{(Turb)}, \tilde{u}^{(Turb)}\right\}^{(ZF)} \equiv f$$

$$\frac{\partial \tilde{n}^{(ZF)}}{\partial t} + \nu \nabla^4 \tilde{n}^{(ZF)} = -\left\{\tilde{\phi}^{(Turb)}, \tilde{n}^{(Turb)}\right\}^{(ZF)} \equiv g$$

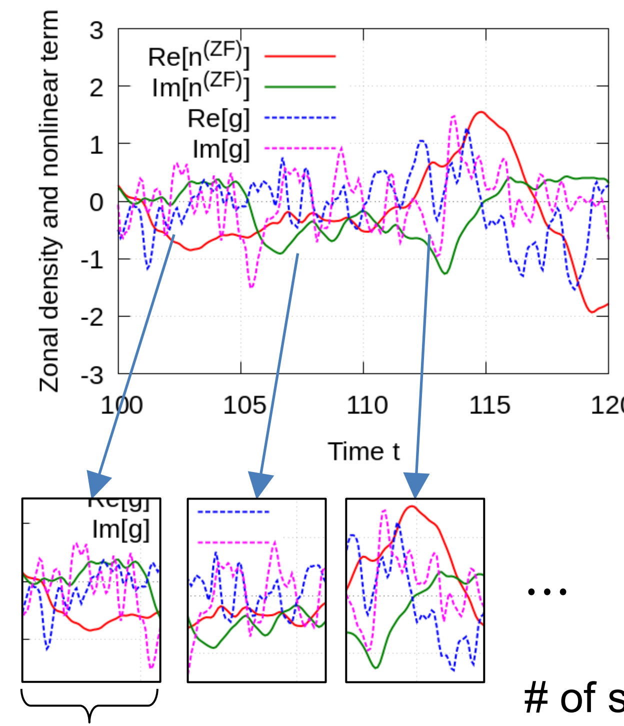
- Turbulent interactions oscillate in time, acting both of driving/damping.

Apply M-Z projection

- Divide the data into Nsample samples of time-series data with length Nperiod
- Statistical average is evaluated by sample average,

$$\langle f(t) u^* \rangle = \frac{1}{N_{\mathrm{sample}}} \sum_{i=0}^{N_{\mathrm{sample}}-1} f^{(i)}(t) u^{*(i)}(0)$$

Time evolution of Fourier coefficient of $\tilde{n}^{(ZF)}$ and turbulent interaction g ($k_x \rho_i = 0.3$)



Length N_{period}

\# of samples $= N_{\mathrm{sample}}$

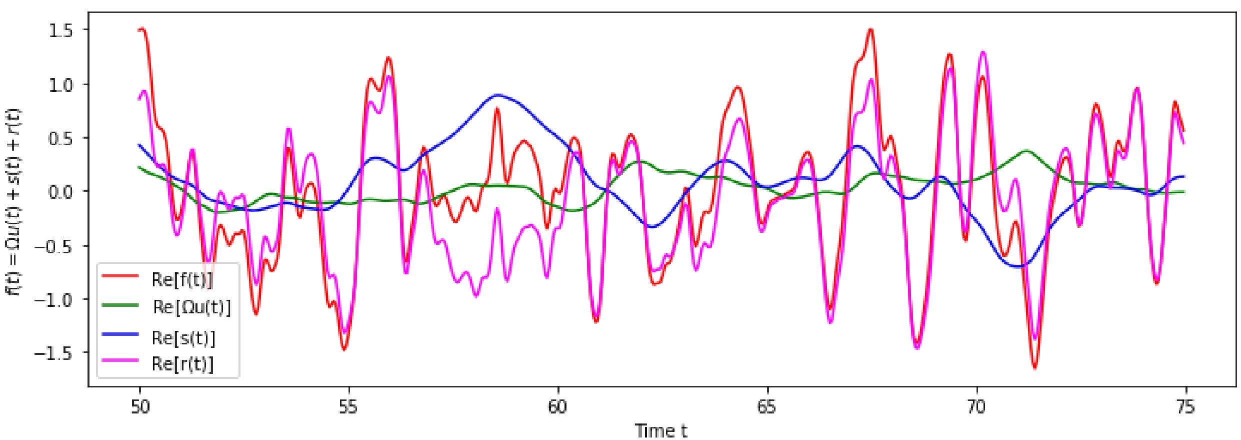
Applying M-Z projection to turbulence-zonal flow interactions

- A time-series data of Reynolds stress $f(t) = -\{\tilde{\phi}^{(Turb)}, \tilde{\omega}^{(Turb)}\}^{(ZF)}$ is split into correlated and uncorrelated terms w.r.t zonal vorticity $\tilde{u}_{zf}(t)$ by M-Z projection.

$$\boldsymbol{f}(t) = \Omega \cdot \boldsymbol{u}(t) + \boldsymbol{M}(t) + \boldsymbol{r}(t), \qquad \text{with } \boldsymbol{M}(t) = -\int_0^t \Gamma(s) \cdot \boldsymbol{u}(t-s) ds$$

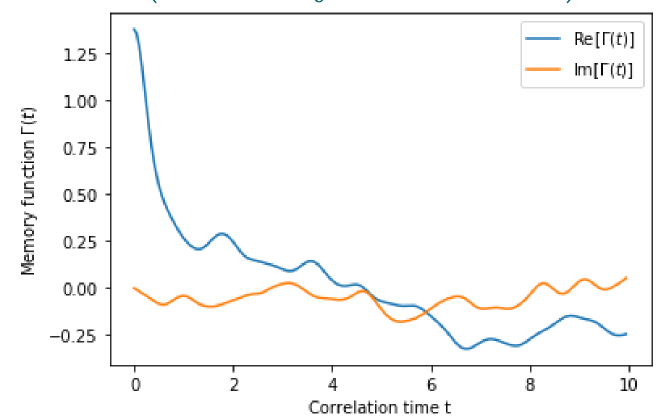
- Memory function $\boldsymbol{\Gamma}(\boldsymbol{t})$ is constructed from DNS data. It typically decays in time scale ~3, faster than oscillation of zonal flows >10.

Time evolution of $\boldsymbol{f_k}(\boldsymbol{t}) = \boldsymbol{\Omega_k} \tilde{\boldsymbol{u}}_{\boldsymbol{k}}(\boldsymbol{t}) + \boldsymbol{M_k}(\boldsymbol{t}) + \boldsymbol{r_k}(\boldsymbol{t})$



Memory function $\boldsymbol{\Gamma}(\boldsymbol{t})$

$$\left(\boldsymbol{M}(t) = -\int_0^t \Gamma(s) \cdot \boldsymbol{u}(t-s) ds\right)$$



Physical insights from M-Z projection

M-Z projection provides a generalized Langevin picture of turbulence effect.

$$\underbrace{\hat{f}_k(t)}_{\text{Reynolds stress}} = \underbrace{\Omega_k \hat{u}_k(t) + \widehat{M}_k(t)}_{\text{Correlated part (~ Mean drag)}} + \underbrace{\hat{r}_k(t)}_{\text{Uncorrelated part (~ Stochastic forcing)}}$$

Evaluated Ω_k, $\Gamma_k(t)$, $\hat{r}_k(t)$ give physical interpretation. For example,

- Complex drag causes,
  $Re[\Omega_k] > 0$: driving
  $Re[\Omega_k] < 0$: damping
  $Im[\Omega_k]$: phase shift

- If r_k only has rapid characteristic time scale θ_k, it may look stochastic.

Wavenumber dependence of M-Z coefficients

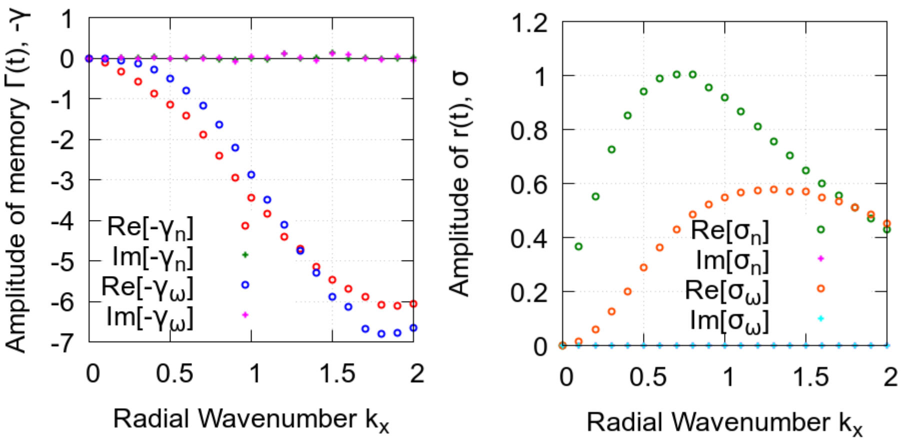
If the memory time of $\Gamma_k(t)$ is short, Markov approximation gives a simple form,

$$\frac{\partial \tilde{u}^{(ZF)}}{\partial t} + \nu \nabla^4 \tilde{u}^{(ZF)} \simeq (\Omega - \gamma)\tilde{u}^{(ZF)} + r(t)$$

From evaluated M-Z coefficients,
- $\mathrm{Re}[\Omega_k - \gamma_k] < 0$
- Rapid oscillation of $r_k(t)$ with variance σ_k^2

→ In H-W turb., turbulence effects on zonal flows are mean damping (by memory term) and stochastic forcing (by uncorrelated term).

Coefficients of mean part $-\gamma_k \propto \hat{M}_k(t)$ and uncorrelated part $\sigma_k \propto \hat{r}_k(t)$



Modeling for reduced simulations

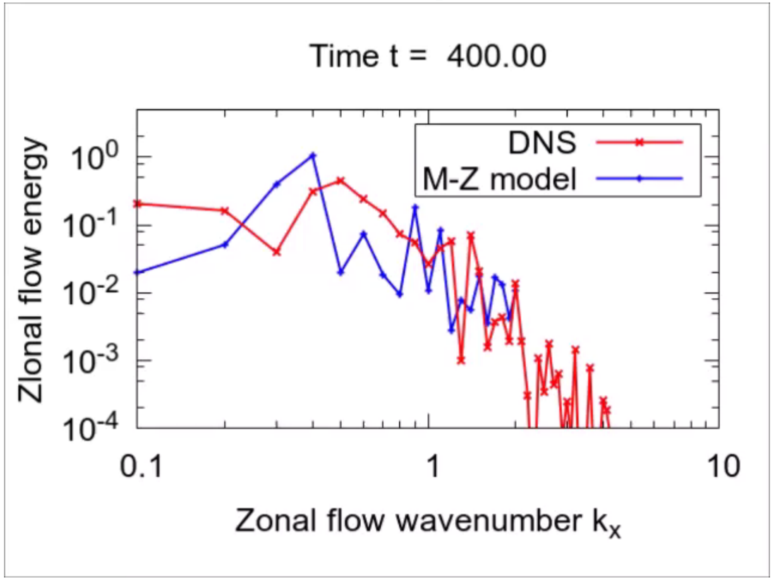
If we approximate $r_k(t)$ by a stochastic process, we can construct a reduced simulation with a model of turbulence effects.

$$\frac{\partial \tilde{u}^{(ZF)}}{\partial t} + \nu \nabla^4 \tilde{u}^{(ZF)} = \sum_k f_k(t) e^{-ikx},$$

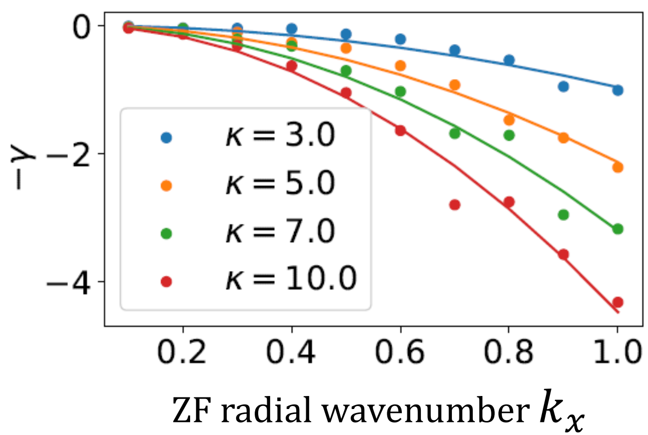
$$\left(\begin{array}{l} f_k(t) = \Omega_k \tilde{u}_k^{(ZF)} + M_k(t) + r_k(t), \\ \dfrac{dM_k(t)}{dt} = -\dfrac{\gamma_k}{\tau_k} \tilde{u}_k^{(zf)} - \dfrac{M_k}{\tau_k}, \\ \dfrac{dr_k}{dt} = -\dfrac{r_k}{\theta_k} - \dfrac{\sigma_k}{\theta_k} W(t) \end{array} \right)$$

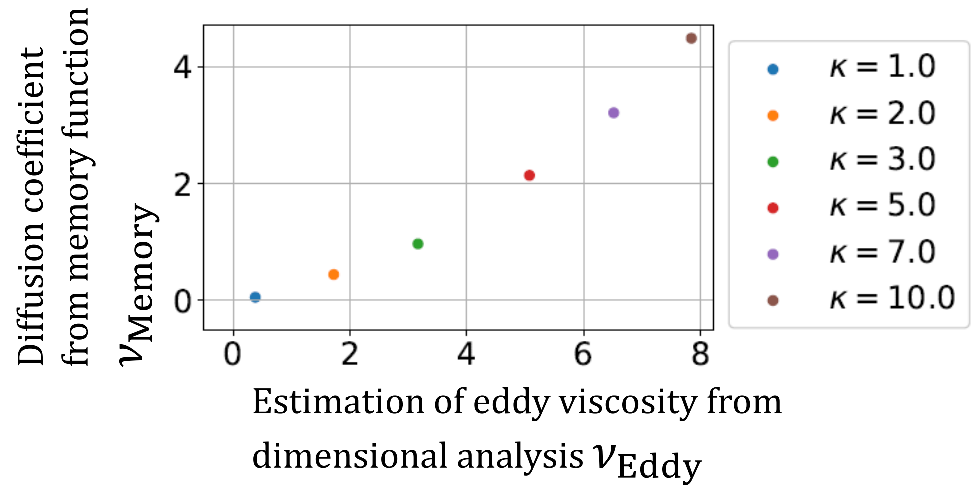
- ✓ Reduced simulation (solving only ZFs, but turbulence effects are modeled) reproduces a spectrum of DNS.
- ✓ It confirms that M-Z projection extracted essential features of turbulence effects.

ZF simulation with a turbulence effect model is compared with DNS.



Dependence of mean damping rate on turbulence intensity （$\propto\kappa$）





（Left）Damping rate by memory term is diffusive (for small k_x): $\gamma \propto \nu_{\mathrm{Memory}} k_x^2$

Assuming it is random walk diffusion by turbulent flows, a rough estimate $\nu_{\mathrm{Eddy}} \sim \frac{1}{\langle k^2 \rangle \langle t \rangle}$ by typical spatial scale $\langle k \rangle^{-1}$ and time scale $\langle t \rangle \sim \langle k \rangle^{-1} \langle v \rangle^{-1}$ of turbulence

（Right）As turbulent intensity（$\propto\kappa$）increases, both of ν_{Eddy} and ν_{Memory} increase. Qualitatively agree.

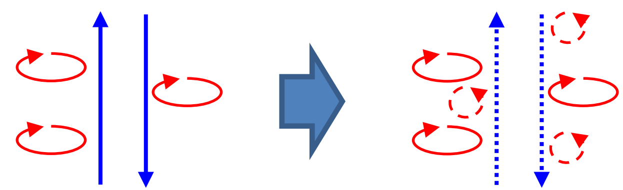
Physical interpretation: Generalized Langevin picture of turbulence-zonal flow interactions

$$\frac{\partial \tilde{u}^{(ZF)}}{\partial t} + \nu \nabla^4 \tilde{u}^{(ZF)} = \underbrace{f(t)}_{\text{Turbulence effects}} = \underbrace{\Omega \tilde{u}^{(ZF)}(t) + M(t)}_{\text{Dissipation by a mean drag}} + \underbrace{r(t)}_{\text{Stochastic forcing by a noise}}$$

In statistically steady state,

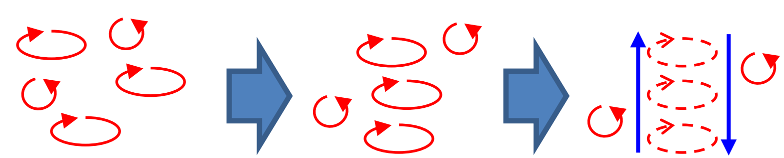
Mean drag
(zonal flow-turbulence interactions)

- Zonal flows tend to be damped in the presence of turbulent flows.



Stochastic forcing
(turbulence-turbulence interactions)

- Turbulence statistically drives (or damps) zonal flows when the phase of turbulent eddies matches.



Summary

- Mori-Zwanzig projection operator method is a statistical tool to analyze correlation among time-series data. [Maeyama, JPSJ 89, 024401 (2020)]

$$\boldsymbol{f}(t) = \Omega \cdot \boldsymbol{u}(t) - \int_0^t \Gamma(s) \cdot \boldsymbol{u}(t-s) ds + \boldsymbol{r}(t)$$

  - You can use it. Download (Fortran90 or Python3) from https://github.com/smaeyama/mzprojection

- M-Z projection is applied to turbulence-zonal flow interactions.
  - It gives physical insights as a generalized Langevin picture.
  - Turbulence effects on zonal flows are mean damping (by memory term) and stochastic forcing (by uncorrelated term).

Failed model: $\frac{\partial \tilde{u}_{zf}}{\partial t} = -\nu k_x^2 \tilde{u}_{zf} - \boldsymbol{\xi}(\boldsymbol{t})$ [a random noise]

Successful model: $\frac{\partial \tilde{u}_{zf}}{\partial t} = -\nu k_x^2 \tilde{u}_{zf} - \boldsymbol{\gamma_k \tilde{u}_{zf}} + \boldsymbol{\xi}(\boldsymbol{t})$ [mean drag and a random noise]

  - Mean damping rate by memory term would be qualitatively understood as an eddy viscosity by random walk process in turbulent flows.