Theory for control of Alfvén instabilities and implications for anomalous electron energy transport

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Motivation and Main Results

- **Motivation**: sub-cyclotron Alfvén Eigenmodes (AEs) have been experimentally linked to anomalous electron temperature flattening in NSTX
  - No theory quantitatively reproduces the observations

- **Goal**: predict instability conditions for realistic neutral beam (NBI) distributions using analytic theory and numerical simulations

- **Main result**: simple theory describes high frequency AE excitation and demonstrates how to stabilize modes with additional NBI source
  - Explains NSTX-U suppression of AEs with new beam source
  - Provides insight to control and study the associated electron energy transport
Outline

- Introduction: Alfvén Eigenmodes Linked to Anomalous Electron Transport
- Hybrid Simulations Reveal Complicated Stability Boundaries
- Simple Analytic Theory Explains Simulations
- Theory Yields Experimental Insights
- Injecting Multiple Beams Can Control Alfvén Eigenmodes
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Anomalous $T_e$ Flattening in NSTX Correlates with CAE/GAEs

- Beam-driven compressional (CAE) and global (GAE) Alfvén eigenmodes have been excited in NSTX(-U), MAST, DIII-D, AUG, and may be present in ITER.
- Temperature profiles cannot be explained by turbulence in gyrokinetic simulations.
- Methods to control CAEs/GAEs are essential to studying and predicting the electron energy transport that they induce.\(^1\)

\(^1\)D. Stutman et al. Phys. Rev. Lett. 102, 115002 (2009)
How Can CAEs/GAEs Affect Temperature Profiles?

**Energy Channeling**
- AE in core can mode convert to KAW near edge, damping on electrons
- Modifies effective beam energy deposition profile\(^2,3\)

**Orbit Stochastization**
- Sufficiently many unstable AEs can stochasticize electron orbits
- Enhances diffusion, transporting energy away from the core\(^4\)

\[^4\]N.N. Gorelenkov *et al.* Nucl. Fusion **50**, 084012 (2010)
Sub-cyclotron Alfvén Eigenmodes in NSTX(-U)

- **Compressional Alfvén eigenmode (CAE):**
  ideal magnetosonic mode: $\omega \approx kv_A$

- **Global Alfvén eigenmode (GAE):**
  discrete shear Alfvén eigenmode existing below minimum of Alfvén continuum: $\omega \leq \left[ k_\parallel(r)v_A(r) \right]_{\text{min}}$

- CAEs/GAEs interact with fast ions through Doppler-shifted cyclotron resonance
  $$\omega - k_\parallel \langle v_\parallel \rangle = \ell \langle \omega_{ci} \rangle$$

- Observed to propagate both **co-** ($k_\parallel > 0$, $\ell = 0$) and **cntr-** ($k_\parallel < 0$, $\ell = 1$) to the beam/plasma current with $|n| = 3 - 15$, $\omega/\omega_{ci} \approx 0.1 - 1.2$
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Hybrid Simulation Method

- Hybrid MHD and Particle code (HYM)
- Initial value code in tokamak geometry
- Thermal plasma: single fluid resistive MHD model
- Beam ions: full orbit kinetic particles with $\delta f$ scheme
  - Captures Doppler-shifted cyclotron resonance which drives the modes
- Equilibrium includes fast ion effects self-consistently\(^5\)
- Thermal plasma and beam ions coupled through current in momentum equation

\[
\rho \frac{dV}{dt} = -\nabla P + (J - J_b) \times B - e n_b (E - \eta \delta J) + \mu \Delta V
\]

\[^5\text{E.V. Belova et al. Phys. Plasmas 10, 3240 (2003)}\]
Fast Ion Distribution Model

- Equilibrium distribution $F_0 = F_1(v)F_2(\lambda)F_3(p_\phi)$
  - Trapping parameter $\lambda = \mu B_0/\mathcal{E} \approx v_\perp^2/v^2$
    
    \[ F_1(v) = \frac{1}{v^3 + v_c^3} \]
    
    \[ F_2(\lambda) = \exp\left(-\frac{(\lambda - \lambda_0)^2}{\Delta \lambda^2}\right) \]
    
    \[ F_3(p_\phi) = \left(\frac{p_\phi - p_{\min}}{m_i R_0 v - q_i \psi_0 - p_{\min}}\right)^\sigma \]
    for $v < v_0$ and $p_\phi > p_{\min}$

- NSTX: $v_0/v_A \lesssim 5$, $\lambda_0 = 0.5 - 0.7$ for original beam
- NSTX-U: $v_0/v_A \lesssim 2$, $\lambda_0 = 0$ for new beam
- Parameters matched to TRANSP ($\Delta \lambda \approx 0.3$)
Hybrid Simulations Predict Rich Mixture of CAEs/GAEs

- Parameter scan of injection geometry ($\lambda_0$) and velocity ($v_0/v_A$) reveals complicated stability boundaries for different mode types$^6$
  - Simulated $|n| = 1 - 12$ separately
- GAEs excited at lower beam energy ($v_0/v_A \gtrsim 2.5$) than CAEs ($v_0/v_A \gtrsim 4$), typically with larger growth rates
- co-GAEs excited with very tangential beams
  - Anomalous cyclotron resonance ($\ell = -1$)
  - May exist in future NSTX-U experiments

$^6$J.B. Lestz et al. Nucl. Fusion 61, 086016 (2021)
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\(^6\)J.B. Lestz \textit{et al.} Nucl. Fusion \textbf{61}, 086016 (2021)
Realistic Simulations Motivate Development of Theory

- HYM simulations accurately model CAEs/GAEs in NSTX(-U) experiments and recover mode structures from CAE3B eigensolver.
- Theory is needed to interpret simulation results and improve understanding to develop predictive capability.

![Diagram showing HYM simulations and experiment results]
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Simple Conditions Derived for Net Drive

• Fast ion drive depends on gradients of the distribution

\[ \gamma \propto \int h(\lambda, \nu) \left[ \left( \ell \frac{\omega_{ci}}{\omega} - \lambda \right) \frac{\partial}{\partial \lambda} + \frac{\nu}{2} \frac{\partial}{\partial \nu} \right] f_b(\lambda, \nu) d^2\nu > 0 \text{ for instability} \]

• cntr-propagating CAE/GAEs (\(\ell = +1\)) driven by \(\partial f_b/\partial \lambda > 0\)
  - \(v_0 < v_{||,\text{res}} / (1 - \lambda_0 \langle \omega_{ci} \rangle)^{3/4}\) necessary for instability

• co-propagating CAEs (\(\ell = 0\)) driven by \(\partial f_b/\partial \lambda < 0\)
  - \(v_0 > v_{||,\text{res}} / \left(1 - \frac{\langle \omega_{ci} \rangle}{2} \left[ \lambda_0 + \sqrt{\lambda_0^2 + 8\Delta\lambda^2/3} \right] \right)^{5/8}\) necessary for instability

• Due to resonance condition, \(v_{||,\text{res}}(\omega/\omega_{ci}, k_\parallel/k_\perp) = (\omega - \ell \langle \omega_{ci} \rangle) / k_\parallel \)

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Analytic Bounds Explain Simulation Results

- NUMERICALLY INTEGRATE FULL ANALYTIC EXPRESSION FOR GROWTH RATE TO PREDICT INSTABILITY
  - red: net fast ion drive, blue: net fast ion damping
  - gray: insufficient beam velocity for resonant interaction

\[ \text{cntr-GAE } \omega / \omega_{ci} = 0.2 \]

\[ \text{predicted stable} \]

\[ \text{predicted unstable} \]

\[ \gamma / \omega_{ci} = 0.08 \]

\[ \text{co-CAE } \omega / \omega_{ci} = 0.5 \]

\[ \text{predicted stable} \]

\[ \text{predicted unstable} \]
Analytic Bounds Explain Simulation Results

- Numerically integrate full analytic expression for growth rate to predict instability
  - red: net fast ion drive, blue: net fast ion damping
  - gray: insufficient beam velocity for resonant interaction

- black curve: approximate analytic conditions reproduce numerical calculation
Analytic Bounds Explain Simulation Results

• Numerically integrate full analytic expression for growth rate to predict instability
  – red: net fast ion drive, blue: net fast ion damping
  – gray: insufficient beam velocity for resonant interaction

• black curve: approximate analytic conditions reproduce numerical calculation

• gold: unstable modes from HYM simulations agree with theory
Beam Parameters Determine Most Unstable Modes

- cntr-GAEs prefer $\lambda_0 \to 1$, whereas co-GAEs require small $\lambda_0$ for instability
  - Driven by opposite sign of $\partial f_0/\partial \lambda$
- co-CAEs are less unstable due to a smaller coefficient multiplying growth rate
  - $\gamma_{\ell=0} \sim (\omega/\omega_{ci})\gamma_{\ell=\pm 1}$
- cntr-GAEs can be destabilized at small $v_0/v_A$
  - co-GAEs require large Doppler shift
  - co-CAEs suffer relatively large $\partial f_0/\partial \nu$
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For fixed NBI parameters, instability conditions constrain the spectrum of modes. Cross-comparison with NSTX database of cntr-GAEs and co-CAEs demonstrates greater than 80% agreement with theory. Blue: NSTX observations, red: HYM simulations. Gray: unstable region predicted by theory.
Theory Explains Previous Observations on DIII-D

- DIII-D low field experiments ($v_0/v_A \approx 1.5$) observed cntr-modes with $\omega/\omega_{ci} \approx 0.6$
  - Tentatively identified as CAEs, with unexplained $k_\perp \rho_\perp b < 0.8$
  - Density scaling also not Alfvénic, conflicting with dispersion relation

- Theory predicts narrow range of unstable GAEs:

\[
1 + v_0/v_A < \omega/\omega_{ci} < 1 + v_0/v_A \left(1 - \lambda_0\right)^{3/4}
\]

- Compatible with any value of $k_\perp \rho_\perp b$
- Weaker density scaling agrees with data

- Theory predicts higher frequencies for larger $\lambda_0$
- Unstable CAEs would require much higher than observed frequencies
- High frequency observations in DIII-D are more consistent with GAEs

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  - Tentatively identified as CAEs, with unexplained $k_\perp \rho_\perp b < 0.8$
  - Density scaling also not Alfvénic, conflicting with dispersion relation\(^9\)

- Theory predicts narrow range of unstable GAEs:
  \[
  \frac{1}{1+v_0/v_A} < \frac{\omega}{\omega_{ci}} < \frac{1}{1+v_0/v_A(1-\lambda_0)^{3/4}}
  \]
  - Compatible with any value of $k_\perp \rho_\perp b$
  - Weaker density scaling agrees with data

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GAE Stabilized with Beam Density Ramp in DIII-D

- cntr-GAE driven by sub-Alfvénic beam ($v_0/v_A = 0.8$) in DIII-D
- Beam density ramped down at constant voltage – mode vanishes\(^\text{10}\)
  - Variable beam perveance\(^\text{11}\)
- Mode identified with dispersion, resonance, and instability condition
  - Theory predicts unstable GAEs have $0.5 < \omega/\omega_{ci} < 0.8$
    - Observed mode: $\omega/\omega_{ci} = 0.58$
- HYM simulations confirmed unstable GAEs with $|n| = 22 - 24$, $m \approx 3 - 4$

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NSTX-U found robust suppression of cntr-GAEs with addition of new off-axis/tangential beams\textsuperscript{12}

Analytic Theory Explains cntr-GAE Stabilization on NSTX-U

• Tangential injection flips sign of $\partial f_b/\partial \lambda \rightarrow$ damping

• **Stabilization**: damping from new beam balances drive from original beam
  
  – 7% of fast ions in new beam predicted for complete stabilization of cntr-GAEs
    
    □ Very close to experiment and HYM simulations$^{13}$

• Surprisingly, simulations show that tangential injection also stabilizes co-CAEs
  
  – Requires $\sim$ 25% fast ions in new beam
  
  – Theory predicts that very perpendicular injection should also stabilize co-CAEs, challenging to verify

Stabilization Techniques for Studying Electron Transport

1. Add a new beam in a different geometry (increase damping from $\partial f_b / \partial \lambda$)
   - To suppress cntr-GAEs/CAEs, add a more tangential beam
     - explains NSTX-U GAE suppression observations
   - To suppress co-CAEs, add a very tangential or perpendicular beam
     - driven by $\partial f_b / \partial \lambda < 0$, opposite condition for cntr-CAEs/GAEs
     - near marginal stability, large radiative damping is sensitive to beam distribution
   - To suppress either, counter-inject a new beam
     - Accesses new resonance for same mode, with opposite contribution to drive

2. Add a new beam at a different voltage without changing geometry
   - Adding a beam at a lower voltage should suppress co-CAEs

3. Add resonant particles which are stabilizing ($\lambda > \lambda_0$ for cntr-GAEs)
   - Can be achieved by lengthening the tail of the distribution – RF heating?
Summary

- CAEs/GAEs were investigated in NSTX(-U) with hybrid simulations and theory
- A simple analytic theory of sub-cyclotron Alfvén eigenmode instability has been developed for realistic NBI distributions
  - **Perpendicular injection**: drives cntr-propagating CAEs/GAEs and damps co-modes
  - **Tangential injection**: damps cntr-CAEs/GAEs and can drive or damp co-modes
  - Explains experimental observations and simulations of CAE/GAE excitation and stabilization in multiple devices (NSTX, NSTX-U, DIII-D)
- **Impact**: theory for control of CAEs/GAEs will enable investigation of their role in electron energy transport and help identify transport mechanisms
- **Future Applications**: (1) project to ITER ($\alpha$ distribution, multiple ion species), (2) try similar approach to interpret ion cyclotron emission (ICE), and (3) model sub-cyclotron instabilities driven by runaway electrons\(^\text{14}\)

\(^{14}\)C. Liu et al. Nucl. Fusion 61, 036011 (2021)
Compressional Alfvén Eigenmodes (CAE)

- Ideal magnetosonic mode in toroidal geometry
  - Compressional polarization
  - In uniform, low $\beta$ limit, $\omega = k v_A$
- Localized by 2D wave equation
  \[
  \left[ \nabla_\perp^2 - V_{\text{eff}}(r, \theta) \right] \delta B_\parallel = 0
  \]
  \[
  V_{\text{eff}}(r, \theta) = k_\parallel^2 - \frac{\omega^2}{v_A^2} \approx \left( \frac{n}{R} \right)^2 - \left( \frac{\omega}{v_A} \right)^2
  \]
- $V_{\text{eff}} = 0$ coincides with Alfvén resonance, where CAE couples to kinetic Alfvén wave

$n = 4$ CAE calculated by HYM. $\delta B_\parallel$ corresponds to the CAE. Coherent $\delta B_\perp$, $\delta E_\parallel$ structures show the KAW.
Global Alfvén Eigenmodes (GAE)

- Discrete shear Alfvén eigenmode solutions may exist below minimum of Alfvén continuum
  - Approximate dispersion $\omega \leq \left[ k_\parallel (r) v_A (r) \right]_{\text{min}}$
  - Weakly damped due to separation from continuum
- Dominant shear polarization: $\delta B_\perp \gg \delta B_\parallel$
  - In NSTX conditions, also have large compressional component $\delta B_\parallel \approx \delta B_\perp$ near edge
- CAEs/GAEs routinely observed in NSTX with $|n| = 3 - 12$ and $\omega / \omega_{ci} \approx 0.1 - 1.2$
  - ICE with $\omega > \omega_{ci}$ also present

GAE calculated by NOVA
**HYM Physics Model**

**Fluid Thermal Plasma**

\[
\rho \frac{dV}{dt} = -\nabla P + (J - J_b) \times B - e n_b (E - \eta \delta J) + \mu \Delta V
\]

\[
E = -V \times B + \eta \delta J
\]

\[
\frac{\partial B}{\partial t} = -\nabla \times E
\]

\[
\mu_0 J = \nabla \times B
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho V)
\]

\[
\frac{d}{dt} \left( \frac{P}{\rho^\gamma} \right) = 0
\]

- \(\rho, V, P\) are plasma mass density, velocity, and pressure
- \(n_b, J_b\) are beam ion density and current \((n_b \ll n_e, \text{though } J_b \approx J_{th})\)

**Kinetic Fast Ions**

\[
\frac{dx}{dt} = v
\]

\[
\frac{dv}{dt} = \frac{q_i}{m_i} (E - \eta \delta J + v \times B)
\]

\[\delta f \text{ Scheme}\]

\[
F = F_0(\varepsilon, \mu, p, \phi) + \delta f(t)
\]

\[
w \equiv \frac{\delta f}{F}
\]

\[
\frac{dw}{dt} = -(1 - w) \frac{d \ln F_0}{dt}
\]
HYM Provides Input for Energy Transport Calculations

- Theories of CAE/GAE-induced electron energy transport require assumptions about the mode properties (frequency, amplitude, polarization, structure, etc.)
- HYM simulations generate realistic mode structures, beyond ideal MHD

![Images of CAE and KAW modes](image)

![Images of cntr-GAEs modes](image)
cntr-GAE Mode Structure from HYM

cntr-GAE  \( n = 6 \), \( \lambda = 0.9 \), \( v = 5.0 \)
co-GAE Mode Structure from HYM

\[ \delta B_{||} \]

\[ \delta B_{\perp} \]

\( n = 8, \lambda = 0.1, \nu = 5.3 \)
Linear Simulation Stability Results

![Diagram showing growth rate vs. injection velocity for different values of n, with co-CAE, cntr-GAE, and co-GAE markers.]
• Local linear growth rate derived for realistic NBI distribution
  – Uncovers new instability regime – necessary to explain GAE excitation in NSTX-U
• **Goal**: simple expressions for fast ion drive depending on
  1. fast ion distribution parameters \((\lambda_0, v_0/v_A)\)
  2. mode parameters \((\omega/\omega_{ci}, k_{||}/k_{\perp})\)
• Approach: restrict to 2D velocity space to avoid assumptions about
  equilibrium profiles, mode structure, particle orbits, etc.
  – Does not include contribution from \(\partial f_0/\partial p_\phi\)
• Provides upper bound on net growth rate, since neglecting bulk damping sources
  – Reminder: \(\gamma_{net} = \gamma_{EP} - \gamma_{th,damp}\)
Growth Rate Calculated for Anisotropic Beam Distribution

- For a beam-like distribution, with $x \equiv v_{\perp}^2/v^2$, $u \equiv v^2/v_0^2 = v_{\parallel, \text{res}}^2/v_0^2(1 - x)$

\[
\frac{\gamma}{\omega_{ci}} \propto - \sum_{\ell} \int_{0}^{1-x} \frac{x}{(1-x)^2} \mathcal{J}_\ell \left( \frac{k_{\parallel} v_{\parallel, \text{res}}}{\omega_{ci}} \sqrt{\frac{x}{1-x}} \right) e^{-\left( x - \lambda_0 \langle \bar{\omega}_{ci} \rangle \right)^2 / \Delta \lambda^2 \langle \bar{\omega}_{ci} \rangle^2} \frac{1 + (4u)^{3/2}}{\left( \frac{2}{\Delta \lambda^2 \langle \bar{\omega}_{ci} \rangle^2} \left( x - \lambda_0 \langle \bar{\omega}_{ci} \rangle \right) \left( \frac{\ell}{\bar{\omega}} - x \right) + \frac{3}{2} \left( \frac{1}{1 + (4u)^{3/2}} \right) \right)} dx
\]

- Can integrate numerically, but further analytic progress requires approximation
Approximations Necessary to Derive Instability Conditions

1. Growth rate dominated by anisotropy for $\ell \neq 0$ resonances

\[
\frac{\omega_{ci}}{\omega} \frac{\partial f_0}{\partial \lambda} \gg \frac{v}{2} \frac{\partial f_0}{\partial v}
\]

2. “Wide beam approximation” for $\Delta x \approx 0.3$

\[
\frac{d}{dx} \frac{e^{-(x-x_0)^2/\Delta x^2}}{\Delta x^2} \approx -2(x - x_0)/\Delta x^2
\]

3. Small (or large) $k_\perp \rho_\perp b$ expansion of finite Larmor radius Bessel function terms

4. Neglect slowing down velocity dependence (weak dependence)
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1. Growth rate dominated by anisotropy for $\ell \neq 0$ resonances

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4. Neglect slowing down velocity dependence (weak dependence)

Define $\eta = v_{||,\text{res}}^2 / v_0^2$, then for $k \rho_b \lesssim 1$ and $\ell = 1$, the growth rate is proportional to

$$\gamma \propto - \int_0^{1-\eta} \frac{x(x-x_0)}{(1-x)^2} \, dx > 0 \quad \Rightarrow \quad x_0 > \frac{1 - \eta^2 + 2\eta \log \eta}{1 - \eta + \eta \log \eta}$$
Approximations Necessary to Derive Instability Conditions

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Define $\eta = v_{\parallel, \text{res}}^2 / v_0^2$, then for $k_{\perp} \rho_{\perp} b \lesssim 1$ and $\ell = 1$, the growth rate is proportional to

\[ \gamma \propto - \int_0^{1-\eta} \frac{x(x-x_0)}{(1-x)^2} \, dx > 0 \longrightarrow x_0 > \frac{1 - \eta^2 + 2\eta \log \eta}{1 - \eta + \eta \log \eta} \approx 1 - \eta^{2/3} \]

1% error for $0 < \eta < 1$

\[ \Rightarrow v_0 < \frac{v_{\parallel, \text{res}}}{(1 - x_0)^{3/4}} \]
Serendipitous Approximations

- Where did this approximation come from?

\[ f(x) = \frac{1 - x^2 + 2x \log x}{1 - x + x \log x} \approx 1 - x^{2/3} \]

Accurate on \( 0 < x < 1 \) to within 1%!

- Assume \( f(x) \approx 1 - x^p \)
  - Preserves smoothness, convexity, and monotonicity
  - \( f(0) = 1 \) and \( f^{(n)}(0) \to (-1)^n \infty \) for \( 0 < p < 1 \)
  - \( f(1) = 0 \) and match \( f'(1) = -p \)

- Correct boundary behavior + sufficiently smooth function \( \longrightarrow \) accurate global approximation
  - Same procedure used many times in this work
Gradients in $p_\phi$ Destabilize Co-Propagating Modes

- Local theory analysis neglected $\partial f_0 / \partial p_\phi$
- Effect can be determined heuristically by comparing to nonlocal theory\textsuperscript{15}

$$ \gamma \propto \int d\Gamma \left[ \left( \frac{l}{\bar{\omega}} - \lambda \right) \frac{\partial f_0}{\partial \lambda} + \mathcal{E} \frac{\partial f_0}{\partial \mathcal{E}} + \frac{n}{\bar{\omega} \omega_{ci}} \frac{\partial f_0}{\partial p_\phi} \right] $$

- For non-hollow distributions, $\partial f_0 / \partial p_\phi > 0$
  \rightarrow sign of $n$ determines contribution
  - co-modes are driven, cntr-modes are damped
- HYM simulations that artificially remove $\partial f_0 / \partial p_\phi$ contribution confirm its effect for co- vs cntr-GAEs

\textsuperscript{15}A.N. Kaufman \textit{et al}. Phys. Fluids \textbf{15}, 1063 (1972)
CAE/GAE Coupling Can Alter Most Unstable Modes

- Including two fluid effects, dispersions are coupled, modifying the polarization

\[
\left[ 1 - \frac{k^2 v_A^2}{\omega^2} \left( 1 - \frac{\omega^2}{\omega_{ci}^2} \right) \right] \left[ 1 - \frac{k^2 v_A^2}{\omega^2} \left( 1 - \frac{\omega^2}{\omega_{ci}^2} \right) \right] = \frac{\omega^2}{\omega_{ci}^2}
\]

- Changes growth rate, most unstable parts of spectrum
  - Most important for cntr-CAEs, also co-CAEs at smaller \( v_0/v_A \)

- \( \ell = 0 \) co-GAE can not exist without this coupling
Method for Inferring $k_\parallel/k_\perp$ for CAEs

• Measured quantities: $\omega$, $k_\phi$, and $v_A$
• Approximate CAE dispersion relation $\omega \approx k v_A \rightarrow$ infer $k$
• Assume $k_\parallel \approx k_\phi = n/R$ (motivated by mode structure in simulations)
• Derive $k_\perp = \sqrt{k^2 - k_\parallel^2}$ from the above quantities
• Construct $k_\parallel/k_\perp$, which the CAE instability condition depends on
  – Dispersion: $\omega \approx k v_A$ + resonance: $\omega = k_\parallel v_{||,\text{res}} \rightarrow v_{||,\text{res}}/v_A \approx k/k_\parallel$
  – $\ell = 0$ co-CAE instability requires $v_{||,\text{res}}$ sufficiently small $\rightarrow$ lower bound on $k_{||}/k_\perp$
  – Upper bound on $k_{||}/k_\perp$ is derived heuristically, based on largest radial wavelength that will fit within the minor radius
• Beam density scan in simulations shows
  $\gamma_{\text{damp}} / \gamma_{\text{drive}} \approx 20 - 60\%$
• Attributed to continuum/radiative damping since it is insensitive to viscosity and resistivity
• Electron damping (absent in simulations) calculated analytically for unstable modes
  – GAE electron damping rates are very small
    $\gamma_{\text{damp}} / \gamma_{\text{drive}} \sim 1\%$
  – CAE electron damping could be large enough to stabilize some modes near marginal stability
CAEs/GAEs May Be Present in Burning Plasmas

- ITER will have super-Alfvénic NBI and alpha particles ($v_0/v_A = 1.5 - 2$)
- Anisotropy of alphas near the edge could destabilize cntr-GAEs/CAEs
  - Similar to NSTX(-U) beam parameters
- ITER NBI distribution has $\lambda_0 = 0.3 - 0.8$ depending on radius
  - Could be either destabilizing or stabilizing (NSTX-U multi-beam suppression)

- Open question: if the modes are excited, will the anomalous electron transport also be present in ITER or is it unique to spherical tokamaks?
CAE/GAE-Induced Ion Heating Was Also Explored on NSTX

- Anomalously high $T_i > T_e$ was observed in some NBI-dominated NSTX discharges\textsuperscript{16}
- Proof-of-principle stochastic heating of ions by CAEs shown in test particle simulations\textsuperscript{17}
- Subsequent experimental analysis\textsuperscript{18} found CAE to thermal ion power transfer to be insufficient to explain the surplus $T_i - T_e$
- Not yet fully resolved

\textsuperscript{17} N.N. Gorelenkov \textit{et al.} Nucl. Fusion \textbf{43}, 228 (2003)
Open Questions

1. Which transport mechanism is the dominant cause of anomalous flat $T_e$ profiles?
2. Will CAEs/GAEs be unstable in ITER? Will they induce anomalous transport?
3. What is the dominant mechanism for co-CAE stabilization by tangential injection?
4. Can the analytic stability boundaries be generalized to $\omega \gg \omega_{ci}$ in order to interpret ion cyclotron emission (ICE)?