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Compact Equations for 3D Plasma Equilibrium

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ABSTRACT

- The system of equations describing an arbitrary static three-dimensional equilibrium of plasma in a magnetic field with toroidal magnetic surfaces is obtained.
- The system consists of two magnetic differential equations (MDEs) and of an analogue of the Grad–Shafranov equation, which allows one a simple transition to the limiting case of axial symmetry.
- Some approaches and examples of its solution are given.

BACKGROUND

- The Grad–Shafranov equation (GSE)

$$\Delta^*\Psi + 4\pi r^2 \frac{dp(\Psi)}{d\Psi} 1 + \frac{1}{2}\frac{dF^2(\Psi)}{d\Psi} = 0 \qquad \text{GSE}$$

is the main tool for the calculation of axisymmetric equilibrium magnetic configurations with nested magnetic surfaces, which are the basis of magnetic plasma confinement. In GSE, $\Psi(r,z)$ is the poloidal flux function labeling the magnetic surface, $p(\Psi)$ is the plasma pressure, $F(\Psi)$ is the poloidal current function, and Δ^* is the Grad-Shafranov operator.

- The compact form of GSE is a result of the mixed representation of the magnetic field

$$\mathbf{B} = [\nabla\Psi \times \nabla\varphi] + F(\Psi)\nabla\varphi,$$

which automatically satisfies the solenoidality of the magnetic field and magnetic surface conditions in axisymmetric case ($\partial/\partial\varphi = 0$).

- However, it fails to describe the three-dimensional magnetic surfaces with $\Psi = \Psi(r, \varphi, z)$. We modify it to provide an extension for a 3D case.

METHOD

BASIC EQUILIBRIUM EQUATIONS

- Static force balance equation: $\nabla p = \frac{1}{4\pi}[\mathrm{curl}\,\mathbf{B} \times \mathbf{B}]$,
- Solenoidality condition: $\mathrm{div}\,\mathbf{B} = 0$,
- Requirement of the magnetic surface existence: $\mathbf{B}\cdot\nabla\Psi = 0$.

MAGNETIC FIELD REPRESENTATION

$$\mathbf{B} = \boxed{\gamma}[\nabla\Psi \times \nabla\varphi] + F\nabla\varphi - \boxed{\alpha F\nabla\Psi}$$

$$\alpha = \frac{1}{|\nabla\Psi|^2}\frac{1}{r^2}\frac{\partial\Psi}{\partial\varphi},$$

provides the fulfillment of $\mathbf{B}\cdot\nabla\Psi = 0$

gives extra freedom to describe arbitrary vector field

- Extended representation contains three free functions γ, F, and Ψ, which is one more than in the axisymmetric case.
- Condition $\mathbf{B}\cdot\nabla\Psi = 0$ is satisfied identically.
- The plasma pressure is a function of magnetic surface, $p = p(\Psi)$.
- To specify γ, F, and Ψ we use: **1)** $\mathrm{div}\,\mathbf{B} = 0$, **2)** $\mathrm{curl}\,\mathbf{B}\cdot\nabla\Psi = 0$, **3)** projection of the force balance equation on $\nabla\Psi$.

OUTCOME

3D EQUILIBRIUM EQUATIONS:

$$\gamma^2\Delta^*\Psi + \gamma(\nabla_0\Psi\cdot\nabla\gamma) + 4\pi r^2\frac{dp}{d\Psi} + \frac{F}{|\nabla\Psi|^2}(\nabla_0\Psi\cdot\nabla F) + \frac{1}{\alpha|\nabla\Psi|^2}\left(\mathbf{B}\cdot\nabla\left(F\alpha\frac{\partial\Psi}{\partial\varphi}\right) - \frac{F^2}{2r^2}\frac{\partial}{\partial\varphi}\left(\alpha\frac{\partial\Psi}{\partial\varphi}\right)\right) = 0, \qquad \text{analogue of GSE}$$

$$\mathbf{B}\cdot\nabla\frac{\gamma|\nabla\Psi|^2}{F} = [\nabla\Psi\times\nabla\varphi]\cdot\nabla\left(F + \frac{\gamma^2|\nabla\Psi|^2}{F}\right), \qquad \text{MDE 1}$$

$$\mathbf{B}\cdot\nabla F^\star = 4\pi\frac{\partial}{\partial\varphi}\left(p + \frac{B^2}{8\pi}\right). \qquad \text{MDE 2}$$

Notations: $\Delta^* = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$, $\nabla_0 = \nabla - \frac{\partial\Psi}{\partial\varphi}\nabla\varphi$, $F^\star = B_\varphi r = F\left(1 - \alpha\frac{\partial\Psi}{\partial\varphi}\right)$.

- Axisymmetric limit $\partial/\partial\varphi = 0$: analogue of GSE ⇨ GSE

MDE 1 ⇨ $\gamma = \gamma(\Psi)$ MDE 2 ⇨ $F = F(\Psi)$

WEAK ASYMMETRY CASE

$\Psi(r,\varphi,z) = \Psi_0(r,z) + \varepsilon\Psi_1(r,\varphi,z)$,

$F(r,\varphi,z) = F_0(\Psi_0) + \varepsilon F_1(r,\varphi,z)$,

$\gamma(r,\varphi,z) = 1 + \varepsilon\gamma_1(r,\varphi,z)$.

Ψ_0 – a solution satisfying axisymmetric GSE (at $\varepsilon \to 0$)

$\varepsilon \ll 1$ – degree of asymmetry

Method of the solution: step 1 – choose Ψ_0;

step 2 – calculate Ψ_1, F_1, γ_1 from the linearized equations.

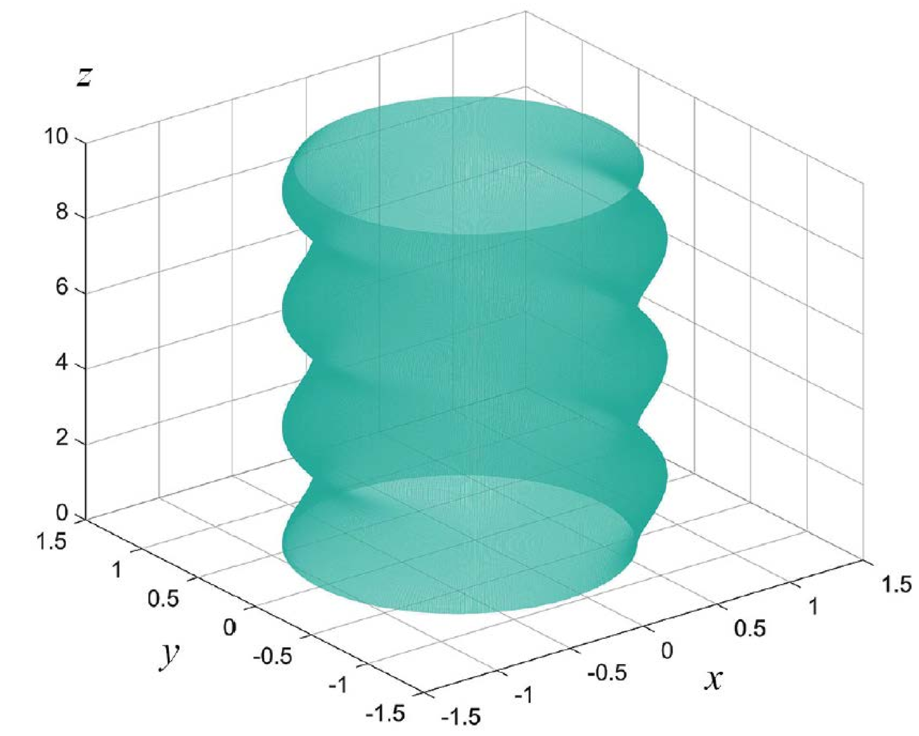
- **Ribbed θ-pinch**: $\Psi_0 = \frac{\Psi_a r^4}{R^4}$ – basic solution to the GSE with $F_0 = 0$, $(dp/d\Psi)_0 = -A/4\pi$

Infinite (along z-axis) θ-pinch, with linearly growing current

3D solution:

$$\Psi = \frac{\Psi_a r^4}{R^4} + \varepsilon\Psi_k(r)\cos kz\cdot\cos n\varphi,$$

$$(k^2r^2 + n^2)\frac{\partial^2\Psi_k}{\partial r^2} - \frac{(k^2r^2 - n^2)}{r}\frac{\partial\Psi_k}{\partial r} - \frac{\Psi_k}{r^2}\left((k^2r^2+n^2)^2 + 4n^2\right) = 0.$$



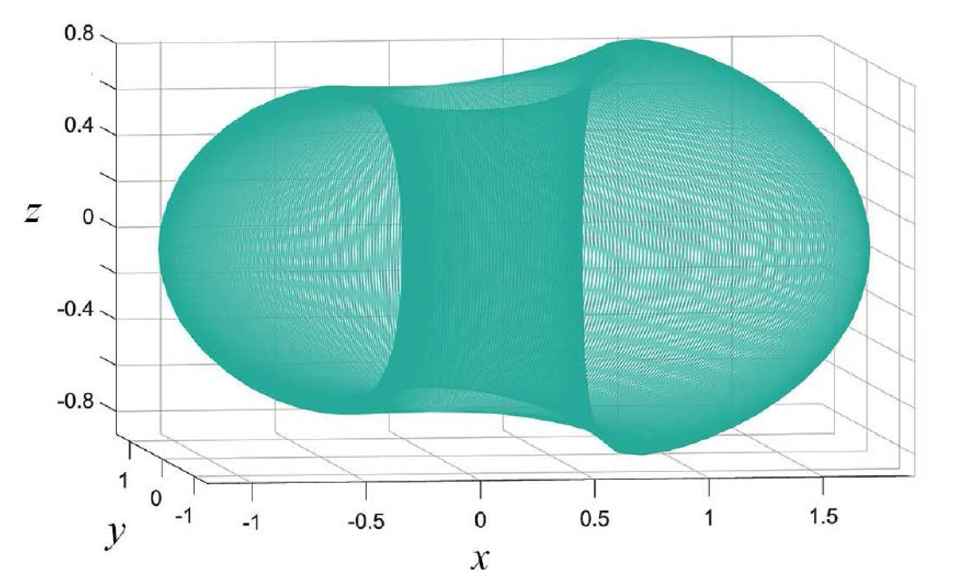
Example of 3D cylindrical mag. surf.

- **Asymmetric Hill vortex:** $\Psi_0 = \frac{\Psi_a r^2}{R^4}(2R^2 - r^2 - 4\lambda^2 z^2)$

Closed toroidal configuration of the Hill vortex type

3D solution:

$$\Psi = \frac{\Psi_a}{R^4}(2R^2r^2 - r^4 - 4\lambda^2r^2z^2 - 4\varepsilon R r(R^2 - r^2 - 2\lambda^2 z^2)\cos\varphi).$$



Example of 3D toroidal mag. surf.

CONCLUSION

- Extended mixed representation of the magnetic field results in a system of three coupled partial differential equations for 3D plasma equilibrium description. One of them is the analogue of GSE and two remaining equations have the form of MDE.
- A stiff connection between the variation of the poloidal current function $F^* = rB_\varphi$ on the magnetic surface and the toroidal asymmetry of the total pressure of plasma and magnetic field is demonstrated. In particular $F^* \neq F^*(\Psi)$ in a 3D case.
- The obtained equations admit a simple and intuitive algorithm for finding solutions, like the GSE, at least in the case of weak axial asymmetry.