The system of equations describing an arbitrary static three-dimensional analogue of GSE is obtained.

The system consists of two magnetic differential equations (MDEs) and of an analogue of the Grad–Shafranov equation, which allows one a simple transition to the limiting case of axial symmetry.

Some approaches and examples of its solution are given.

The compact form of GSE is a result of the mixed representation of the magnetic field, which automatically satisfies the solenoidality of the magnetic field and is the main tool for the calculation of axisymmetric equilibrium magnetic configurations with nested magnetic surfaces, which are the basis of magnetic plasma confinement. In GSE, \( \Psi(r, z) \) is the poloidal flux function labeling the magnetic surface, \( p(\Psi) \) is the plasma pressure, \( F(\Psi) \) is the poloidal current function, and \( \Delta^* \) is the Grad-Shafranov operator.

However, it fails to describe the three-dimensional magnetic surfaces with \( \Psi = \Psi(r, \varphi, z) \). We modify it to provide an extension for a 3D case.

The compact form of GSE is a result of the mixed representation of the magnetic field

\[
B = [\nabla \Psi \times \nabla \varphi] + F(\Psi) \nabla \varphi,
\]

which automatically satisfies the solenoidality of the magnetic field and magnetic surface conditions in axisymmetric case (\( \partial / \partial \varphi = 0 \)).

3D EQUILIBRIUM EQUATIONS:

\[
\gamma^2 \Delta^* \Psi + \gamma (\nabla \Psi \cdot \nabla) \Psi + 4 \pi r^2 \frac{dp}{d\Psi} + \frac{F}{|\nabla \Psi|^2} (\nabla \Psi \cdot \nabla F) + \frac{1}{\alpha |\nabla \Psi|^2} (B \cdot \nabla \left( \frac{\partial \Psi}{\partial \varphi} \right) - F^2 \frac{\partial}{\partial \varphi} \left( \frac{\partial \Psi}{\partial \varphi} \right)) = 0.
\]

\[
B \cdot \nabla |\nabla \Psi|^2 = [\nabla \Psi \times \nabla \varphi] \cdot \nabla \left( F + \frac{1}{\gamma} \frac{|\nabla \Psi|^2}{F} \right).
\]

\[
B \cdot \nabla F^* = 4 \pi \frac{\partial}{\partial \varphi} \left( p + B^2 / \gamma \right),
\]

Notations: \( \Delta^* = \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \varphi^2} \), \( \nabla \varphi = \nabla - \frac{\partial \Psi}{\partial \varphi} \nabla \varphi \), \( F^* = B_{\varphi \varphi} + F \left( 1 - \frac{\partial \Psi}{\partial \varphi} \right) \).

The obtained equations admit a simple and intuitive algorithm for finding solutions, like the GSE, at least in the case of weak axial asymmetry.

Static force balance equation:

\[ p = \frac{1}{4 \pi} \text{curl} \left[ \frac{\partial}{\partial \varphi} \left( \nabla \Psi \times \nabla \varphi \right) \right] \]

Solenoidality condition:

\[ \text{div} B = 0, \]

Requirement of the magnetic surface existence:

\[ B \cdot \nabla \Psi = 0, \]

MDE 1

\[ \gamma = \gamma(\Psi) \]

MDE 2

\[ F = F(\Psi) \]

WEAK ASYMMETRY CASE

\( \Psi(r, \varphi, z) = \Psi_0(r, \varphi, z) + \epsilon \Psi_1(r, \varphi, z) \)

\( \Psi_0 \) – a solution satisfying axisymmetric GSE (at \( \varepsilon \to 0 \)), \( \Psi_1 \) – a solution satisfying axisymmetric GSE (at \( \varepsilon \to 0 \))

Method of the solution: step 1 – choose \( \Psi_0 \), step 2 – calculate \( \Psi_1, F_1, \gamma_1 \) from the linearized equations.

Ribbed \( \theta \)–pinch: \( \Psi_0 = \frac{p_r r^4}{R^4} \) – basic solution to the GSE with \( F_0 = 0 \), \( (dp/d\Psi)_0 = -A/4\pi \)

\( \Psi = \frac{p_r r^4}{R^4} + \epsilon \Psi_k(\varphi \cos k \varphi \sin \varphi, \sin k \varphi \cos \varphi) \)

\begin{align*}
(\kappa^2 r^2 + n^2) \frac{\partial^2 \Psi_k}{\partial r^2} &- \left( k^2 r^2 - n^2 \right) \frac{\partial \Psi_k}{\partial r} \\
&- \frac{\Psi_k}{r^2} \left( (k^2 r^2 + n^2)^2 + 4n^2 \right) = 0.
\end{align*}

Asymmetric Hill vortex:

\( \Psi_0 = \frac{p_r r^4}{R^4} \left( 2R^2 - r^2 - 4p^2 z^2 \right) \)

\( \Psi = \frac{p_r r^4}{R^4} \left( 2R^2 - r^2 - 4p^2 z^2 - 4 \epsilon \epsilon_{r r} (R^2 - r^2 - 2\lambda^2 z^2) \cos \varphi \right) \)

Example of 3D toroidal mag. surf.

Example of 3D toroidal mag. surf.

CONCLUSION

Extended mixed representation of the magnetic field results in a system of three coupled partial differential equations for 3D plasma equilibrium description. One of them is the analogue of GSE and two remaining equations have the form of MDE.

A stiff connection between the variation of the poloidal current function \( F^{\ast} = r B_{\varphi \varphi} \) on the magnetic surface and the toroidal asymmetry of the total pressure of plasma and magnetic field is demonstrated. In particular, \( F^{\ast} + F(\Psi) \) in a 3D case.