Progress in theoretical understanding of the Dimits shift and the tertiary instability of drift waves

Hongxuan Zhu
Coworkers: Yao Zhou and I. Y. Dodin

Princeton Plasma Physics Lab and Princeton University

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Drift waves are low-frequency waves in magnetized plasmas

They can be excited by as simple as a density gradient and a phase difference between N and Φ.

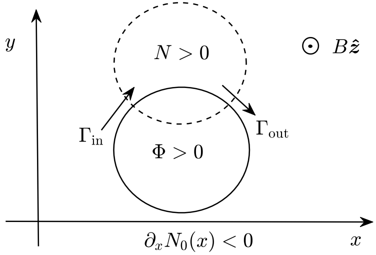
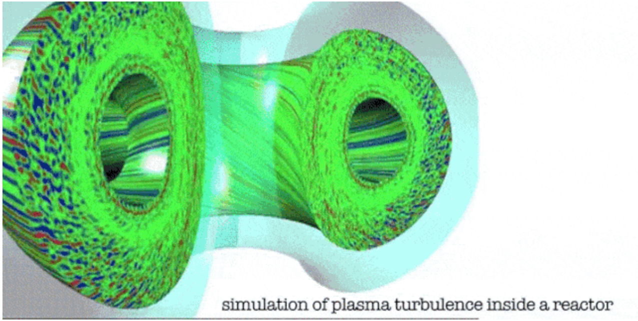


Figure 1: $\boldsymbol{B} = B\hat{\boldsymbol{z}}$: magnetic field, $N_0(x)$: equilibrium density, x: radial direction, y: poloidal direction, Φ: electrostatic potential perturbation, N: density perturbation, Γ: particle flux from $\boldsymbol{E} \times \boldsymbol{B}$ drift.

Drift waves can also be excited by temperature gradient ($N \to T$). A famous example being ion-temperature-gradient (ITG) mode.

Zonal flows are axisymmetric but radially band-like shear flows

They can greatly reduce transport level due to the special electron response.[1]



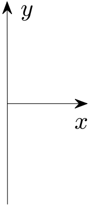


Figure 2: Zonal flows in tokamaks, from GYRO simulations by Jeff Candy and Ron Waltz. Movie from `w3.pppl.gov/~hammett/viz/viz.html`.

[1] G. Hammett et al. Plasma Phys. Control. Fusion **35**, 973 (1993).

Zonal flows can even completely suppress drift waves, leading to the Dimits shift

The Dimits shift is a shift between the threshold of linear instabilities (without zonal flows) and the actual turbulent onset (with zonal flows).

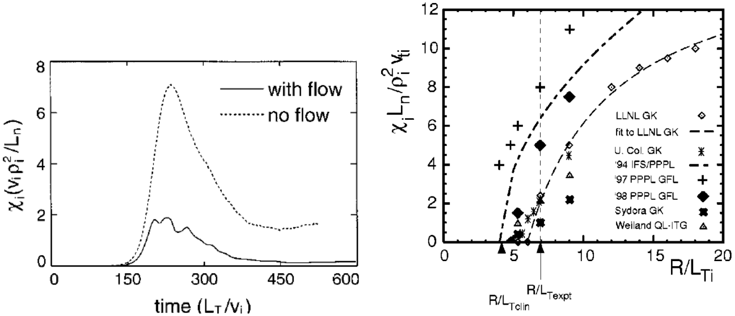


Figure 3: Left: reduced ion heat flux due to zonal flows [Lin *et al.*, Science (1998)]. Right: ITG-driven turbulent transport. The threshold is shifted due to zonal flows. [Dimits *et al.* Phys. Plasmas (2000).]

Conventional theory cannot explain the Dimits shift

The shear decorrelation theory[1] assumes homogeneous drift-wave turbulence. However, in the "Dimits regime", zonal flows dominate and drift waves are inhomogeneous.

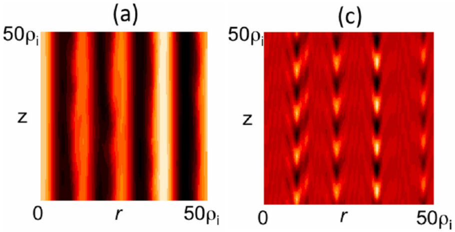


Figure 4: GS2 simulations of entropy modes in Z pinches. Left: total electrostatic potential. Right: nonzonal part of the potential. [S. Kobayashi and R. Rogers, Phys. Plasmas **19**, 012315 (2012).]

[1] H. Biglari, P. H. Diamond, and P. W. Terry, Phys. Fluids. B **2**, 1 (1990).

The “tertiary instability” was proposed to explain the Dimits shift

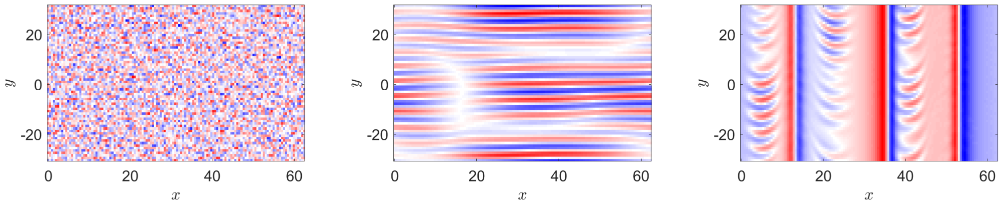


Figure 5: The primary instability (drift-wave generation), the secondary instability (zonal-flow generation), and the tertiary instability. From simulations of the modified Terry–Horton model (introduced below).

The Dimits regime ends when zonal flows are unstable to the “tertiary instability”.

- This is first demonstrated by gyrokinetic toroidal ITG simulations.[1]

However, understanding of the tertiary instability remains elusive.

- Several papers calculate the tertiary instability by making truncation in Fourier space.
- Many have confused the tertiary instability with the Kelvin–Helmholtz instability.
- Need to study the tertiary instability and the Dimits shift using simple models.

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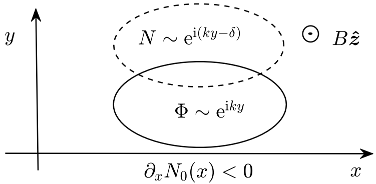
[1] B. N. Rogers, W. Dorland, and M. Kotschenreuther, Phys. Rev. Lett. **85**, 5336 (2000).

Drift-wave dynamics in a 2D slab

- Assume $T_\mathrm{i} = 0$, $T_\mathrm{e} \neq 0$. Define dimensionless quantities

$$n_\mathrm{e} \doteq \frac{a}{\rho_\mathrm{s}} \frac{N_\mathrm{e}}{N_0}, \quad \varphi \doteq \frac{a}{\rho_\mathrm{s}} \frac{e\Phi}{T_e}, \quad (x, y) \to \frac{(x, y)}{\rho_\mathrm{s}}, \quad t \to \frac{c_\mathrm{s} t}{a},$$

where $c_\mathrm{s} = \sqrt{T_\mathrm{e}/m_\mathrm{i}}$, $\rho_\mathrm{s} \doteq c_\mathrm{s}/\Omega_\mathrm{i}$, and a is a reference length.



- Ion guiding-center continuity equation: $\partial_t n_\mathrm{i} + \boldsymbol{v} \cdot \nabla(N_0 + n_\mathrm{i}) = 0$, with $\boldsymbol{v} \doteq \hat{\boldsymbol{z}} \times \nabla\varphi$.
- Quasineutrality constraint for cold ions: $n_\mathrm{i} - n_\mathrm{e} = -\nabla^2\varphi$.
- Need an equation for the electron dynamics.

The modified Terry–Horton equation (MTHE, one-field model)

- The Terry–Horton model[1] makes a simple assumption for the electron density:

$$n_{\mathrm{e}} = (1 - \mathrm{i}\hat{\delta})\varphi.$$

Then, the ion continuity equation and the quasineutrality condition give

$$\partial_t w + \boldsymbol{v} \cdot \nabla w - \kappa \partial_y \varphi + \hat{D} w = 0, \quad w \doteq (\nabla^2 - 1 + \mathrm{i}\hat{\delta})\varphi.$$

Here $\kappa \doteq a/L_n$, $L_n \doteq -(\partial_x \ln N_0)^{-1}$, and $\hat{D}$ is friction/viscosity.

- St-Onge proposed the MTHE[2] with special electron response to zonal flows:[3]

$$n_{\mathrm{e}} = (1 - \mathrm{i}\hat{\delta})\varphi \quad \to \quad n_{\mathrm{e}} = (1 - \mathrm{i}\hat{\delta})\tilde{\varphi},$$

where $\tilde{\varphi} \doteq \varphi - \langle \varphi \rangle$ and $\langle \varphi \rangle$ is the zonal average (average over y for the 2D model).

- The MTHE exhibits finite Dimits shift.[2,4]

[1] T. Terry and W. Horton, Phys. Fluids **25**, 491 (1982).
[2] D. A. St-Onge, J. Plasma Phys. **83**, 905830504 (2017).
[3] W. Dorland and G. Hammett, Phys. Fluids B **5**, 812 (1993). G. Hammett et al. PPCF **35**, 973 (1993).
[4] H. Zhu, Y. Zhou, and I. Y. Dodin, J. Plasma Phys. **86**, 905860405 (2020).

The modified Hasegawa–Wakatani equation (MHWE, two-field model)

- The relation $n_\mathrm{e} = (1 - \mathrm{i}\hat{\delta})\varphi$ is often oversimplified. Therefore, the Hasegawa–Wakatani model evolves n_e dynamically through parallel Ohm's law:[1]

$$\nabla_\parallel \Phi + \eta J_\parallel = (T_\mathrm{e}/eN_\mathrm{e})\nabla_\parallel N_\mathrm{e},$$

  where η is the resistivity. This leads to the HWE:

$$(\partial_t + \hat{\boldsymbol{z}} \times \nabla\varphi \cdot \nabla)w = \kappa\partial_y\varphi - \hat{D}w, \quad w \doteq \nabla^2\varphi - n,$$
$$(\partial_t + \hat{\boldsymbol{z}} \times \nabla\varphi \cdot \nabla)n = \alpha(\varphi - n) - \kappa\partial_y\varphi - \hat{D}n.$$

  Here, $n \doteq n_\mathrm{e}$ and $\alpha \propto \eta^{-1}$. $\alpha \to \infty$: adiabatic limit; $\alpha \to 0$: hydrodynamic limit.
- The HWE has been modified to incorporate the special electron response to zonal flows:[2]

$$\alpha(\varphi - n) \quad \to \quad \alpha(\tilde{\varphi} - \tilde{n}).$$

  The MHWE also exhibits Dimits regime and transition to turbulence.[2,3]

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[1] A. Hasegawa and M. Wakatani, Phys. Rev. Lett. **50**, 682 (1983).
[2] R. Numata, R. Ball, and R. L. Dewar, Phys. Plasmas **14**, 102312 (2007).
[3] A "flux-balanced" HWE is also proposed recently [A. Majda, D. Qi , and A. Cerfon, Phys. Plasmas **25**, 102307 (2018)] to study the Dimits shift, which will not be further discussed here.

In the Dimits regime, drift waves are localized where the flow shear U' vanishes

- Here, $U \doteq \langle\varphi\rangle'$ and the prime $'$ denotes $\mathrm{d}/\mathrm{d}x$.
- Drift waves have different structures at the bottom and the top of U.
- These indicate the properties of the tertiary instability.

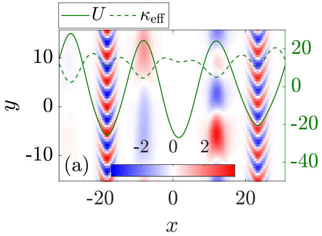


Figure 6: Simulation results from the MHWE with $\alpha = 5$, $\kappa = 12$, $\hat{D} = 0.1\nabla^4$. Green curves show the zonal-flow velocity U and the effective density gradient $\kappa_{\mathrm{eff}} \doteq \kappa - \langle n\rangle'$. Blue and red color show the drift-wave fluctuations $\tilde{w}(x, y)$.

Tertiaty instability of a zonal flow as an eigenmode problem

- Consider the MHWE (the MTHE is even simpler). An arbitrary zonal-flow profile

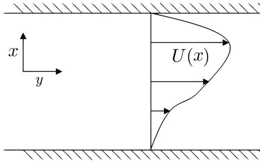
$$U = U(x), \quad \langle n \rangle = 0.$$

  is a stationary solution (assuming small collisional damping).

- Consider a perturbation $\tilde{w} = \mathrm{Re}[\psi(x)\mathrm{e}^{\mathrm{i}(p_y y - \omega t)}]$. The linearized MHWE becomes an eigenmode equation:

$$\omega\psi = \hat{H}\psi, \quad \hat{H} = p_y[U + (\kappa + U'')\hat{\bar{p}}^{-2}] - \mathrm{i}\hat{D},$$

  where $\hat{\bar{p}}^2 \doteq \hat{p}_x^2 + p_y^2 + (\mathrm{i}\alpha + \mathrm{i}\hat{D} + \omega - p_y U)^{-1}(\mathrm{i}\alpha + p_y\kappa)$ and $\hat{p}_x \doteq -\mathrm{id}/\mathrm{d}x$.



- Hydrodynamic limit ($\alpha \to 0$): Continuum modes, boundary layer modes, and Kelvin–Helmholtz instability.
- Plasma ($\alpha \gtrsim 1$): potential well and discrete energy level.

Numerical eigenmodes are obtained for $U = u\cos q_x x$ with periodic boundary condition

For the two most unstable eigenmodes $\psi(x)$, calculate the Wigner function:[1]

$$W(x, p_x) \doteq \int ds\, e^{-ip_x s}\psi(x+s/2)\psi^*(x-s/2).$$

W can be considered as a “distribution function” and peak at equilibria of the Hamiltonian H:

$$\partial_x H = \partial_{p_x} H = 0 \quad \Rightarrow \quad U' = p_x = 0.$$

This explains the mode localization at $U' = 0$. $U_{\max}$: “trapped” mode; $U_{\min}$: “runaway” mode.[2]

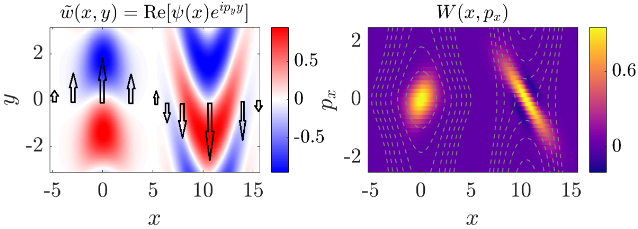


Figure 7: Left: two most unstable eigenmodes plotted together, arrows show zonal-flow velocity. Right: the corresponding W; dashed lines are contours of the Hamiltonian $H = p_y U + p_y(\kappa + U'')\mathrm{Re}\bar{p}^{-2}$.

[1] D. Ruiz, J. Parker, E. Shi, and I. Y. Dodin, Phys. Plasmas **23**, 122304 (2016).
[2] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Plasmas **25**, 072121 (2018).

Analytic eigenmode as quantum harmonic oscillators

Due to the mode localization in x where $U' = 0$, we approximate U as a parabola:

$$U \approx U_0 + \mathcal{C}x^2/2, \quad \mathcal{C} \doteq U''.$$

Assume that mode is localized also in p_x, we expand $\hat{H}$ in both x and $\hat{p}_x$:

$$\hat{H} \approx \mathcal{H} + \lambda_p(\mathcal{C})\,\hat{p}_x^2 + \lambda_x(\mathcal{C})\,x^2.$$

Then, the eigenmode equation $\omega\psi = \hat{H}\psi$ becomes

$$-\lambda_p\,\psi'' + \lambda_x\,x^2\psi = (\omega - \mathcal{H})\psi.$$

Eigenmodes are given by Hermite polynomials H_m, where $m = 0, 1, ...$

$$\psi_m = \mathrm{e}^{-\frac{x^2}{2\lambda}}\mathsf{H}_m(x/\sqrt{\lambda}), \quad \omega_m = \mathcal{H} + (2m+1)\lambda_x\lambda, \quad \lambda = \sqrt{\lambda_p/\lambda_x}.$$

The quantities $\mathcal{H}$, λ_x, and λ_p have been calculated accordingly.[1]

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Rev. Lett. **124**, 055002 (2020).

Numerical vs analytic eigenmodes

Both the trapped mode and the runaway mode are ground states ($m = 0$). They correspond to $\mathcal{C} \doteq U'' < 0$ and $\mathcal{C} > 0$, respectively.

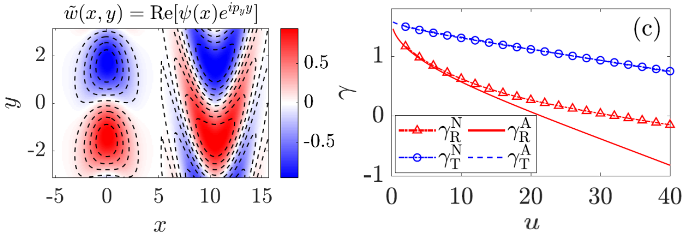


Figure 8: Left: numerical vs analytic mode structures of the MHWE at $\alpha = 5, \kappa = 12, q_x = 0.4, u = 10$. Right: numerical vs analytic growth rate $\gamma_{\mathrm{TI}} \doteq \mathrm{Im}\,\omega_0$.

The tertiary instability shares similar features across different drift-wave models

- For the simple 1-field MTHE, we can go even deeper into analytic calculations.[1]
- Our theory is not restricted to the sinusoidal zonal flows. (Only the local shape of U matters.)

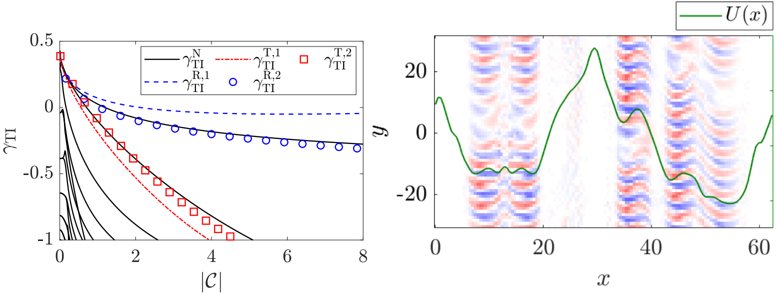


Figure 9: Left: numerical vs analytic tertiary-instability growth rate of the MTHE (recall that $\mathcal{C} = U''$). Right: we can always approximate the local U as a parabola (from MTHE simulations).

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, J. Plasma Phys. **86**, 905860405 (2020).

Properties of the tertiary instability

- Absent zonal flows, $\omega = \mathcal{H}$ is simply the linear drift-wave frequency. Correspondingly, the growth rates have two parts ("PI" stands for the primary instability and $\mathcal{C} \doteq U''$ is the zonal-flow "curvature"):

$$\gamma_{\text{TI}} = \gamma_{\text{PI}} + \Delta\gamma(\mathcal{C}), \quad \Delta\gamma(\mathcal{C} = 0) = 0.$$

  Therefore, the tertiary instability is the primary instability modified by the zonal flow.

- The tertiary instability develops when the zonal flow is weak, and it is suppressed by strong zonal flows. Therefore, the tertiary instability is not the Kelvin–Helmholtz instability.[1,2]

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Rev. Lett. **124**, 055002 (2020).
[2] H. Zhu, Y. Zhou, and I. Y. Dodin, J. Plasma Phys. **86**, 905860405 (2020).

The tertiary instability is not the Kelvin–Helmholtz instability

- The tertiary instability extracts energy from density gradient just like the primary instability. In contrast, the Kelvin–Helmholtz instability extracts energy from the flow shear.
- Due to the special electron response to zonal flows, the Kelvin–Helmholtz instability is usually suppressed.[1] (Unless U is unrealistically large.)

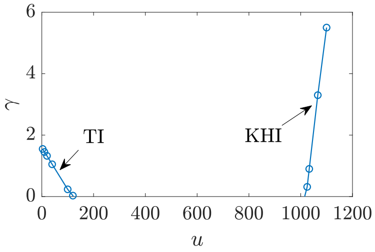


Figure 10: Instability of a zonal flow with $q_x = 0.4$ and varying amplitude u. The model is the MHWE with $\alpha = 5$ and $\kappa = 12$. Only the tertiary instability is relevant for numerical simulations ($u \lesssim 20$).

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Plasmas **25**, 082121 (2018).

Interpretation of the Dimits shift from the tertiary instability

The tertiary-instability growth rate provides a generic understanding of the Dimits shift. Namely, from $\gamma_{\mathrm{TI}} = \gamma_{\mathrm{PI}} + \Delta\gamma(\mathcal{C})$, we have

$$\gamma_{\mathrm{TI}} = 0 \quad \Rightarrow \quad \kappa = \kappa^*,$$
$$\gamma_{\mathrm{PI}} = 0 \quad \Rightarrow \quad \kappa = \kappa_{\mathrm{lin}}.$$

The difference $\Delta_{\mathrm{DS}} \doteq \kappa^* - \kappa_{\mathrm{lin}}$ gives the Dimits shift.

<u>This shift is determined by the flow curvature $\mathcal{C} \doteq U''$, not by the flow shear U'.</u>

Analytic prediction of the Dimits shift of the one-field MTHE model

For the MTHE, Δ_{DS} is calculated analytically, which gives decent agreement with numerical simulations.[1,2]

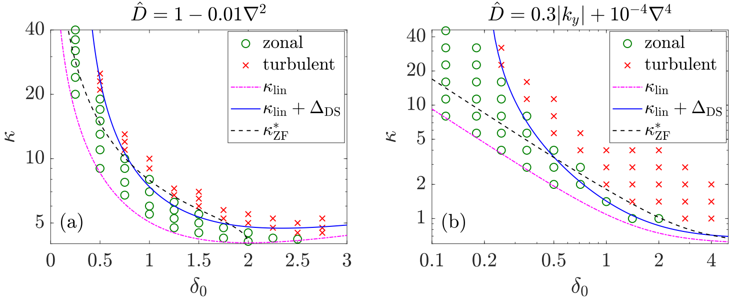


Figure 11: Analytic prediction vs numerical simulation results of the MTHE with $\delta_{\boldsymbol{k}} = \delta_0 p_y$ and two different $\hat{D}$. Also, κ^*_{ZF} is the prediction by St-Onge [J. Plasma Phys. **83**, 905830504 (2017)].

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, J. Plasma Phys. **86**, 905860405 (2020).
[2] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Rev. Lett. **125**, 055002 (2020).

There are more physics related to the Dimits shift of the two-field MHWE model

- Tertiary modes can flatten the local density profile (reducing $\kappa_{\text{eff}} \doteq \kappa - \langle n \rangle'$), which in turn suppresses the tertiary instability. (Predator–prey oscillations.)
- Therefore, the Dimits shift cannot be predicted from analytic formulation assuming $\langle n \rangle = 0$.

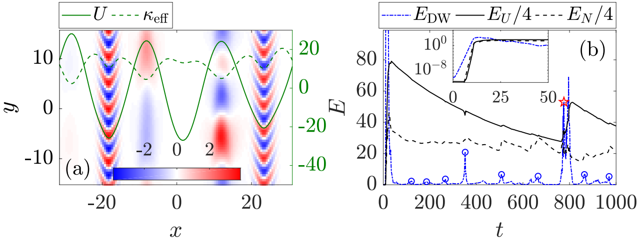


Figure 12: Predator–prey oscillations between E_{DW} and $E_N \doteq \int \langle n \rangle^2 \, d\boldsymbol{x}/2$ (blue circles).

The tertiary instabilities seed turbulent bursts in the MHWE

- Tertiary modes can generate nonlinear propagating structures, triggering turbulent bursts.[1,2,3] Similar structures are seen in a 2D fluid ITG model.[4]
- We expect that the transition from Dimits regime at small κ to turbulent regime at large κ is due to increasingly frequent turbulent bursts.

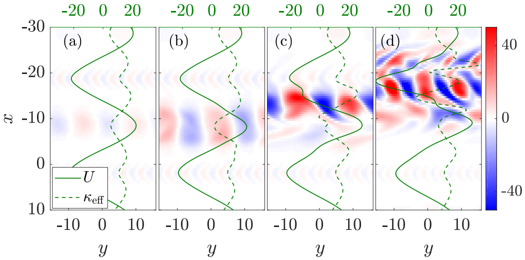


Figure 13: Left: a tertiary mode grows to finite amplitude and generates a turbulent burst.

[1] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Rev. Lett. **125**, 055002 (2020).
[2] Q. Di, A. Majda, and A. Cerfon, Phys. Plasmas **27**, 102304 (2020).
[3] Y. Zhang, S. Krasheninnikov, Plasma Phys. Control. Fusion **62**, 115018 (2020).
[4] P. Ivanov, A. Schekochihin, W. Dorland, A. Field, and F. Parra, J. Plasma Phys. **86**, 855860502 (2020).

Summary

We studied the tertiary instabilities in detail from simple drift-wave models.[1,2]

- Tertiary modes are localized at where $U' = 0$, as visualized using the Wigner function.
- A key finding is that $\gamma_{\mathrm{TI}} = \gamma_{\mathrm{PI}} + \Delta\gamma(\mathcal{C})$, where $\mathcal{C} = U''$ is the local flow curvature.
- Also, the tertiary instability is not the Kelvin–Helmholtz instability.
- These features are model-independent. In fact, we also studied the original gyrofluid ITG model by Rogers *et al.*[3] (See next slide if interested, slides available for download.)

The result $\gamma_{\mathrm{TI}} = \gamma_{\mathrm{PI}} + \Delta\gamma(\mathcal{C})$ provides a generic understanding of the Dimits shift.

- We obtained an analytic prediction of Δ_{DS} within the one-field MTHE model.
- Things are more complicated for the two-filed MHWE model. Nevertheless, transition to turbulent regime is still associated with tertiary instabilities and the nonlinear propagating structures. (See our paper for more discussions.)

Thank you for your attention!

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[1] H. Zhu, Y. Zhou, and I. Y. Dodin, J. Plasma Phys. **86**, 905860405 (2020).
[2] H. Zhu, Y. Zhou, and I. Dodin, Phys. Rev. Lett. **124**, 055002 (2020).
[3] B. N. Rogers, W. Dorland, and M. Kotschenreuther, Phys. Rev. Lett. **85**, 5336 (2000).

Application to the original gyrofluid ITG model by Rogers *et al.*

- The tertiary instability was first discussed from a simplified ITG model:[1]

$$\partial_t n + \hat{\boldsymbol{z}} \times \nabla\phi \cdot \nabla n + \hat{\boldsymbol{z}} \times \nabla(\tau\nabla^2\phi) \cdot \nabla T/2 = 0, \quad \partial_t T + \hat{\boldsymbol{z}} \times \nabla\phi \cdot \nabla T = 0.$$

  Our analytic method is also applicable to this model, leading to $\gamma_{\mathrm{TI}} \sim p_y\sqrt{\tau T' U''}$,[2] consistent with Rogers' original results and a more recent calculation.[3]
- Gyrokinetic simulations show that $\mathcal{C} = U''$ is stabilizing while the $\eta = T'$ is destabilizing. This supports the conclusion that the tertiary instability is *not* the Kelvin–Helmholtz instability.

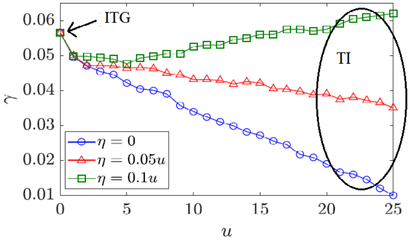

Figure 14: Gyrokinetic GS2 simulation results (slab geometry) with $q_x = 0.2$ and varying u and η.

[1] B. N. Rogers, W. Dorland, and M. Kotschenreuther, Phys. Rev. Lett. **85**, 5336 (2000).
[2] H. Zhu, Y. Zhou, and I. Y. Dodin, Phys. Rev. Lett. **124**, 055002 (2020). Supplemental material.
[3] P. Ivanov, A. Schekochihin, W. Dorland, A. Field, and F. Parra, J. Plasma Phys. **86**, 855860502 (2020).