Topological bifurcations of the magnetic axis and the alternating-hyperbolic sawtooth

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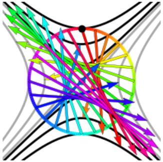
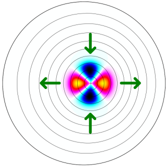
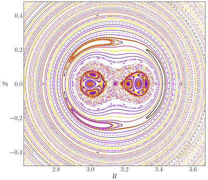




- When $q_0 = 2/3$ the structure of the field can transition to an alternating-hyperbolic configuration.
- In equilibria where $q_0 \sim 2/3$ there is a strong ideal mode located on the axis.
- The displacement associated with this mode drives the topological transition and causes magnetic chaos around the axis.
- Sawteeth observed around $q_0 \sim 0.7$ could thus be explained by this mode when q_0 reaches $2/3$, and the stochastisation drives rapid reconnection that resets the cycle.



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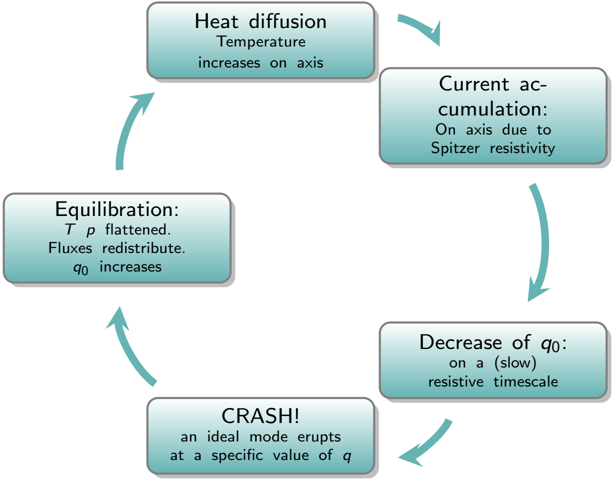
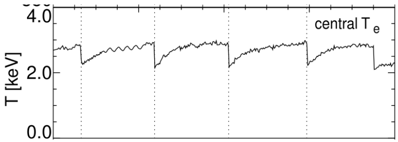
The Sawtooth Oscillation consists of a repeated fast crash

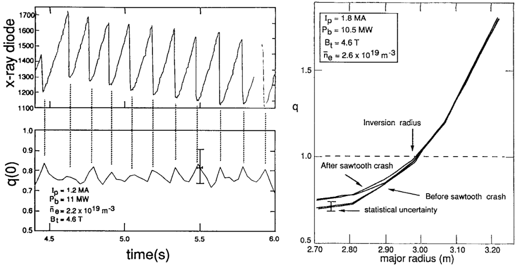




- First observed in 1974.
- Observed in every tokamak built since.
- The precise cause has been debated since 1976.

Many observations do not fit Kadomtsev

- The Kadomtsev model predicts the crash when q_0 is just below 1, and a re-set to $q_0 = 1$; by a 1/1 ideal mode.
- A majority of experiments are consistent with this[1,2,3], but some experiments show a crash occurring around $q_0 = 0.7$.
- This has been observed using different techniques on different machines.
  FR: Faraday Rotation.
  MSE: Motional Stark Effect.
  Li: Lithium Fluorescence imaging.
- The consistency between different devices and measurement techniques suggests a common mechanism for this subset of observations.



Device	q_0 at crash
TEXTOR[a,b] (FR)	0.7 ± 0.1
TFTR[c,d] (MSE)	0.7 ± 0.1
TEXT[e] (Li)	$0.7 - 0.8$
JET[f] (FR & MSE)	$0.7 - 0.8 \pm 0.1$

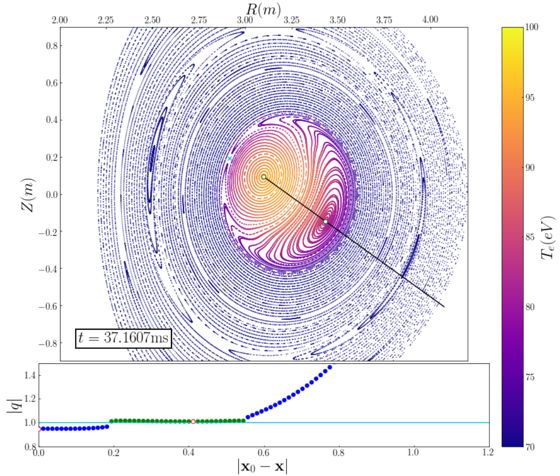
[a] Soltwisch, Rev. Sci. Instrum. 57, 1939 (1986)
[b] Soltwisch, Rev. Sci. Inst. 59, 1599 (1988)
[c] Levinton, F. et al Phys. Fluids B. 5 2554 (1993)
[d] Yamada, Phys. Plasmas 1 3269 (1994)
[e] West, Phys. Rev. Lett. 58 2758 (1987)
[f] Wolf, Nuclear Fusion, 33 663 (1993)

[1] Weller, Phys. Rev. Lett. 59 2303 (1987)
[2] Wroblewski Phys. Fluids B 3, 2877 (1991)
[3] Nam, Y et al, NF 58 066009 (2018)

In the Kadomtsev model the crash is caused by a 1/1 mode

- q_0 reaches lightly below 1, and an intact $q = 1$ surface exists.
- This is unstable to the 1/1 internal kink mode, resistive tearing instability of the $q = 1$ surface, or both[a].
- Hot core plasma reconnects with cold plasma outside the $q = 1$ surface and is deposited in a growing $q/1$ island. After the crash the q-profile is very flat.
- The Kadomtsev model predicts a **crash when** $q_0 \sim 1$, and a **re-set to** $q_0 = 1$ after the crash.
- Experiments often observe exactly this[b,c,d].
- But not always . . .

Kadomtsev-like reconnection in M3DC^1 simulations



[a] Coppi, B., et al. Fizika Plazmy 2 (1976): 961-966.

[b] Weller, Phys. Rev. Lett. 59 2303 (1987)

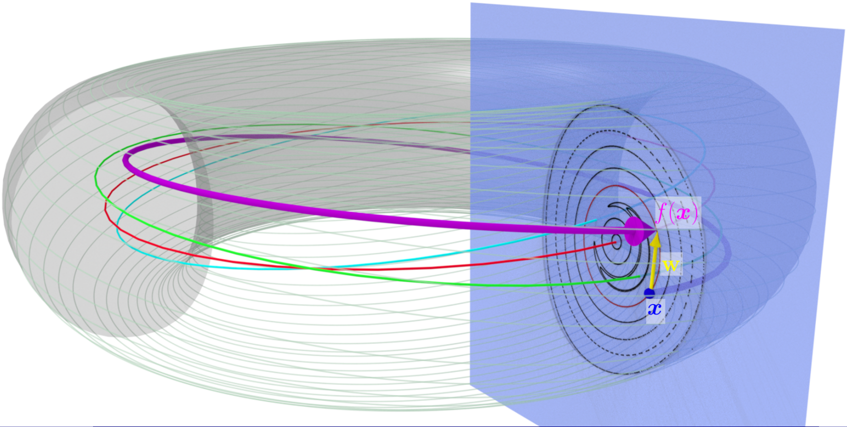
[c] Wroblewski Phys. Fluids B 3, 2877 (1991)

[d] Nam, Y et al, NF 58 066009 (2018)

The structure around the axis is described by a matrix

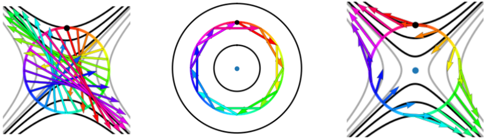
- The magnetic **field line map**; $f(R_0, Z_0) = (R_1, Z_1)^T$ is continuous and differentiable.
- Around a fixed point (f.ex. the magnetic axis), the Jacobian matrix of this mapping M captures the structure of the magnetic field.
- Because $\nabla \cdot \mathrm{B} = 0$ this matrix is area-preserving ($\det(\mathrm{M}) = 1$). $\mathrm{M} \in \mathrm{SL}_2(\mathbb{R})$

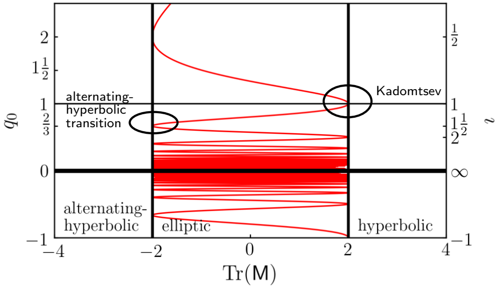
$$\mathrm{M} = \begin{pmatrix} \frac{\partial R_1}{\partial R_0}, & \frac{\partial R_1}{\partial Z_0} \\ \frac{\partial Z_1}{\partial R_0}, & \frac{\partial Z_1}{\partial R_0} \end{pmatrix}$$



The group structure of $\mathrm{SL}_2(\mathbb{R})$ shows all possible configurations

- Through the trace $\mathrm{Tr}(\mathrm{M})$, $\mathrm{SL}_2(\mathbb{R})$ separates into three **conjugacy classes** that act as area-preserving linear transformations of the Euclidean plane: elliptic; O-points, parabolic; surfaces and hyperbolic; X-points
- For **elliptic** points (such as the magnetic axis), the safety factor is related to $\mathrm{Tr}(\mathrm{M})$: $\cos(2\pi/q_0) = \frac{1}{2}\mathrm{Tr}(\mathrm{M})$.
- The continuity of Lie groups is important; the configuration needs an infinitesimal 'push' to change the trace infinitesimally. when $\mathrm{Tr}(\mathrm{M}) = \pm 2$ this push can change the topology
- When $q_0 = 2/3$, an infinitesimal push can turn the axis into an alternating-hyperbolic point. This is very close to the measured $q_0 = 0.7 \pm 0.1$.





Is there an ideal mode that drives this topological transition?

- in order to explain a fast crash, we require a fast ideal mode that forces the topological transition.
- We use the NOVA code to analyze the ideal modes in the core region.
- The NOVA code solves normal mode formulation of the ideal MHD stability equation:

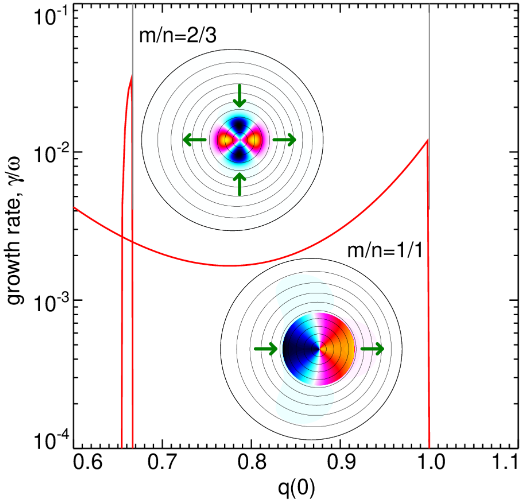
$$-\omega^2 \rho \boldsymbol{\xi} = \boldsymbol{F}[\boldsymbol{\xi}], \tag{1}$$

  where ω^2 is the mode frequency squared, ρ is the plasma density, $\boldsymbol{F}$ is the linerized MHD force operator, and $\boldsymbol{\xi}$ is the displacement vector.
- Solutions $\boldsymbol{\xi}$ with negative ω^2 correspond to ideally unstable, exponentially growing modes.
- The q-profile increases (approximately) quadratically with distance from the axis, has value q_0 at the axis and $2.9 + q_0$ at the plasma edge.

parameter	**value**
$\boldsymbol{B}_T$	$1\mathrm{T}$
R_0	$3\mathrm{m}$
a	$1\mathrm{m}$
$q(\psi)$	$q_0 + 2.9\psi$
β_0	3%

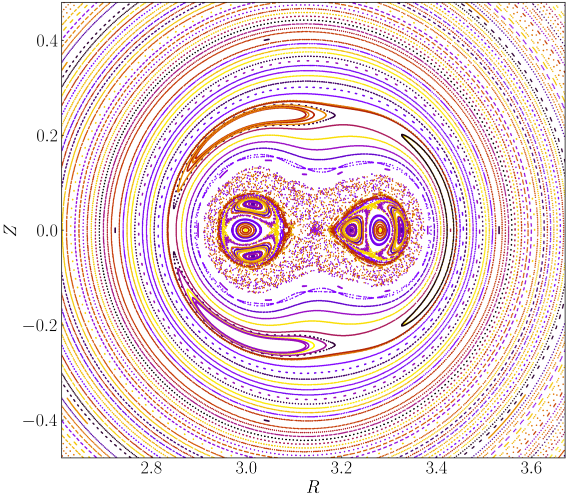
NOVA finds the ideal modes in tokamak-like equilbria

- q_0 is scanned from $q_0 = 1.1$ to $q_0 = 0.6$
- When $q_0 \sim 1$ we see the 1/1 internal kink instability.
- The displacement vector $\boldsymbol{\xi}$ of the 1/1 mode is a rigid translation of the plasma within the $q = 1$ surface. (The internal kink can be stabilized.)
- When $q_0 = 2/3$ we see a new ideal 2/3 instability with an order of magnitude higher growth rate.
- $\boldsymbol{\xi}$ of the 2/3 mode indicates a displacement where the plasma is forced onto the axis in two directions, and away from it in two others.
- This dispacement would moves the magnetic fieldlines closer to the axis in one direction, and other further in another, similar to an alternating-hyperbolic point.



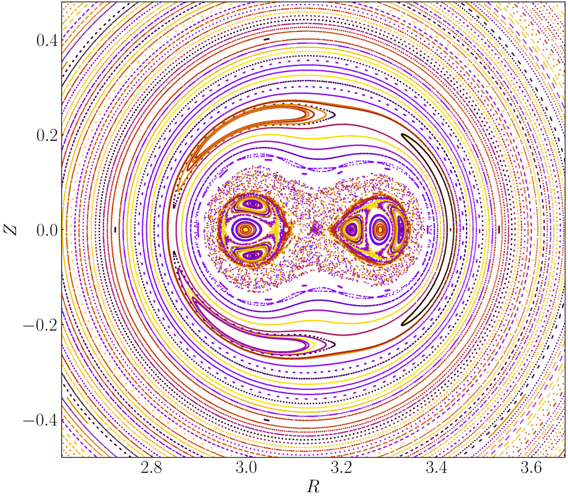
The mode drives the axis to an alternating-hyperbolic point

- We introduce a perturbation corresponding to the ideal displacement of an analytic 2/3 mode located on the axis, on top of the equilibrium with $q_0 = 2/3$.
- At a small amplitude of the perturbation ($|\delta \boldsymbol{B}|/|\boldsymbol{B}_0| \sim 1 \times 10^{-3}$) the axis transitions to the alternating-hyperbolic configuration.
- At higher amplitude of the perturbation ($|\delta \boldsymbol{B}|/|\boldsymbol{B}_0| \sim 2.5 \times 10^{-2}$) a stochastic region is created in the core.
- The $q = 1$ surface is broken up into a 3/3 island chain.



The change in topology will excite other modes

- The change in topology **magnetically connects** the highest-pressure plasma on the axis with surfaces further out.
- Rapid **parallel transport** can induce further perturbations.
- At high amplitude of the perturbation, an entire **stochastic region** is magnetically connected.
- The temperature can **equilibrate** in the connected region.
- Magnetic chaos leads to **rapid reconnection**[a]. This redistributes poloidal and toroidal fluxes, and leads to a new magnetic configuration around the axis with higher q_0



[a] Yi-Min, The Astrophysical Journal 793.2 106 (2014)

The Alternating Hyperbolic sawtooth model

1. We assume that the internal kink is stabilized (If it is not, the Kadomtsev process occurs), and q_0 decreases through slow current diffusion.
2. When q_0 reaches $2/3$, the ideal mode causes the axis to transition to an alternating-hyperbolic fixed point.
3. This mode (possibly with other modes that occur when the topology changes) creates a stochastic region in the core.
4. The temperature equilibrates in the core region, and poloidal and toroidal fluxes are redistributed leading to an increased q_0.
5. The field shifts out of resonance with the $2/3$ mode, the flux surfaces in the core heal, and the cycle begins anew.

End of Poster (extra content below)

The internal kink can be stabilized in many ways

- **Toroidal rotation** stabilizes the internal kink and increases the sawtooth period.[a]
- **Fast particles** from ICRH or neutral beams can stabilize the internal kink and increase the sawtooth period.[b,c]
- **Diamagnetic effects** stabilize the internal kink, and direcly affect observed sawtooth period.[d]
- Any one, or combination of these effects can stabilize the internal kink below $q = 1$, until $q_0 = 2/3$ is reached.

[a] Chapman, Nucl. Fus. 46.12 (2006)

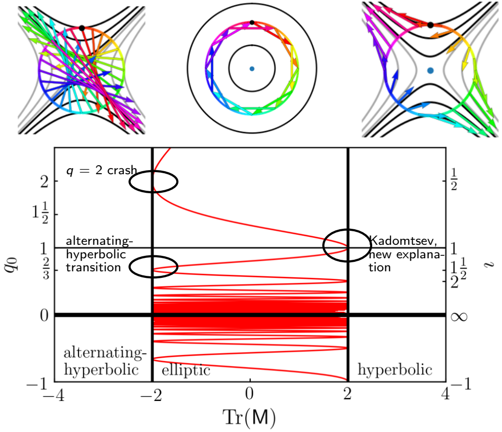
[b] Cole, Phys. Plasmas 21, 072123 (2014)

[c] Porcelli, Plas. Phys. Contr. Fus. 33, 1601 (1991)

[d] Connor, Plas. Phys. Contr. Fus. 54, 035003 (2012)

(Alternaging-)hyperbolic transitions can occur at different values of q_0

- At $q_0 = 2$, sawteeth-like relaxations are observed in reversed-shear profiles[a] (currently attributed to the $q = 2$ double-tearing mode).
- The *new explanation of the sawtooth*[b] by S. Jardin presents simulations of a temperature crash occurring when the 1/1 mode is stabilized, and simulations show a crash due to a 2/2 mode.
- The crashes at $q_0 = 2/3$ have long been observed and debated, and are the topic of this poster
- Transitions at $q_0 = 2/5, \;\; 2/7, \dots$ have not been reported in Tokamaks as they rarely operate with this low q.



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[a] Chang, et al. PRL 77.17 (1996)

[b] Jardin, Phys. Plas. 27.3 (2020).

Jardin's new explanation also involves a topological transition

- Stephen Jardin (et al.) proposed a new explanation of the sawtooth oscillation this year.

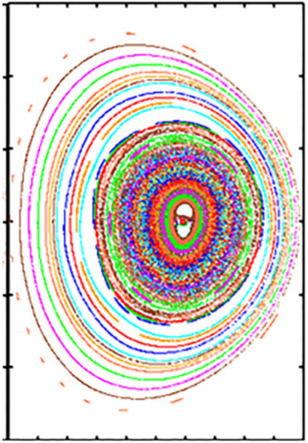
  > *[. . .] a stationary (1, 1) interchange mode will grow and saturate, producing a dynamo voltage that keeps the central q-profile very close to 1 and flat [. . .]*
  > *[. . .] the pressure in the low shear central region will continue to increase until higher-n ideal modes, primarily withm $m = n$, suddenly become unstable causing the central region to become turbulent and stochastic and the temperature and density to flatten.*

  The 2/2 mode is the first to become unstable.

- With $n = 2$, the mode structure of the displacement in the core region is directed towards the magnetic axis from two directions, and away from it it in two other directions.

- When $q_0 = 1$ this can lead to the transition to a hyperbolic fixed point.

- The Poincaré plot of the full nonlinear 3d extended MHD simulation shows that at time of crash there is a 2/2 structure on the axis and a chaotic region.



t = 37.4 ms

Jardin, S. C., Krebs, I., &Ferraro, N. (2020). A new explanation of the sawtooth phenomena in tokamaks. Physics of Plasmas, 27(3), 032509.

Sawteeth observed at $q = 2$ show $2/1$ mode structure and a hyperbolic axis in the simulation

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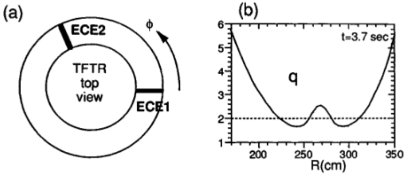
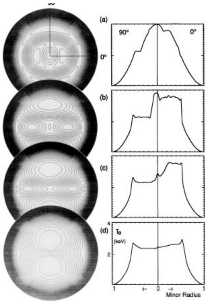
Off-Axis Sawteeth and Double-Tearing Reconnection in Reversed Magnetic Shear Plasmas in TFTR

Z. Chang,[1] W. Park,[1] E. D. Fredrickson,[1] S. H. Batha,[2] M. G. Bell,[1] R. Bell,[1] R. V. Budny,[1] C. E. Bush,[1] A. Janos,[1] F. M. Levinton,[2] K. M. McGuire,[1] H. Park,[1] S. A. Sabbagh,[3] G. L. Schmidt,[1] S. D. Scott,[1] E. J. Synakowski,[1] H. Takahashi,[1] G. Taylor,[1] and M. C. Zarnstorff[1]





Calculating the perturbation

- In the normal mode formulation of the ideal MHD equations, the perturbed field $\delta \boldsymbol{B}$ caused by a displacement $\boldsymbol{\xi}$ is given by:
$$\delta \boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) \tag{2}$$
- We use an analytical perturbation $\boldsymbol{\xi}$ with $\xi_R = -(1/R)\partial_Z \Psi$ and $\xi_Z = (1/R)\partial_R \Psi$.
$$\Psi = A \exp\left(-\frac{\psi_p}{\sigma}\right) \cos(2\theta - 3\phi) \tag{3}$$
- We add this to a Grad-Shafranov equilibrium with $\beta_0 = 3\%$ and $q = 2/3 + 2.3333 * \psi_p$.
- We construct a Poincaré plot of the perturbed field $\boldsymbol{B}_0 + \delta \boldsymbol{B}$ by tracing the orbits of low-energy electrons $(1\mathrm{keV})$ using the SPIRAL code.