Validation of GAE simulation and theory for NSTX(-U) and DIII-D

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Motivation

• Correlation between GAE/CAE observations and flattening of $T_e$ profile at increased NBI power in NSTX suggests that GAE/CAE can reduce the efficiency of NBI heating.
  • Important to correctly identify and predict these instabilities
• Simulations were needed to conclusively identify nature of high-frequency modes ($\omega/\omega_{ci} \sim 0.6$) in DIII-D.
  • Previously identified as compressional Alfven eigenmodes (CAEs) in DIII-D [Heidbrink, NF2006], however both theory and NSTX simulations predict stronger GAE instabilities [Belova PoP2019, Lestz, NF2021].
  • In NSTX(-U), GAEs were more common especially for lower $v_{beam}/v_A$.
  • New dedicated experiments in DIII-D suggest SAW dispersion for these modes [S.Tang, PRL 2021]
• Validate HYM code against DIII-D observations.
• DIII-D results indicate that GAEs can be excited for larger aspect-ratio devices and smaller injection velocity $v_{beam}/v_A \lesssim 1$, therefore they can be unstable in ITER.
Theory and simulations explain GAE frequency scaling with NBI parameters in NSTX, NSTX-U and DIII-D

Numerical model and theory for sub-cyclotron frequency modes developed for NSTX(-U) have been successfully applied to explain DIII-D observations.

- HYM simulations for DIII-D demonstrate that modes with $\omega/\omega_{ci} \sim 0.6$, previously mis-identified as compressional Alfven eigenmodes (CAEs), have shear polarization $\delta B_{\perp} >> \delta B_{||}$ (GAEs).

- Simulation results match the observed frequencies in DIII-D for high- and low- $B_{tor}$ experiments [S.Tang, PRL 2021, Heidbrink NF06].

Scaling of experimentally observed GAE frequency with injection velocity vs predicted by theory (lines) and most unstable modes in simulations (black squares). Color lines – 2-fluid (solid) or MHD (dashed) condition for peak instability calculated for $k_{\perp}/k_{||}=1$. Black lines show $v_{\parallel, res} \leq v_0$ boundary.
Theory predict scaling of most unstable GAE with \( B_{\text{tor}}, n_e, \lambda_0 \)

Predicted range of most unstable counter-GAEs:

\[
\frac{1}{1 + v_0/v_A} < \frac{\omega}{\omega_{ci}} \leq \frac{1}{(1 + \frac{v_0}{v_A})\sqrt{1 - \lambda_0}},
\]

[Belova, PoP 2019]

[B]tor [L estz, PoP 2020]

- \( \omega \sim \omega_{ci} \rightarrow \) nearly linear scaling with \( B_{\text{tor}} \)
- Weaker than \( 1/\sqrt{n_e} \) scaling with density.
- Larger \( v_0/v_A \) results in smaller values of \( \omega/\omega_{ci} \):
  \[
  \begin{align*}
  \omega/\omega_{ci} &\approx 0.6 \text{ for } v_0/v_A \sim 1 \text{ and } \lambda_0 \sim 0.6 \text{ (DIII-D)}, \\
  \omega/\omega_{ci} &\approx 0.4 \text{ for } v_0/v_A \sim 2 \text{ (NSTX-U)}, \\
  \omega/\omega_{ci} &\leq 0.2 \text{ for } v_0/v_A \gtrsim 4 \text{ (NSTX)},
  \end{align*}
  \]
- Scaling with \( \lambda_0 \): larger \( \lambda_0 \rightarrow \) larger \( \omega/\omega_{ci} \)
  \[
  \begin{align*}
  \omega/\omega_{ci} &\approx 0.6 \text{ for } \lambda_0 = 0.5, \\
  \omega/\omega_{ci} &\approx 0.7 \text{ for } \lambda_0 = 0.8 \text{ (} v_0/v_A \sim 1 \text{)}
  \end{align*}
  \]
  – consistent with DIII-D ‘left’ and ‘right’ beam sources \( (\omega/\omega_{ci} = 0.56 \text{ and } \omega/\omega_{ci} = 0.69) \) [Heidbrink,NF06].

\( (\omega/\omega_{ci}) < 1 \) for \( DIII-D \)

\( (\omega/\omega_{ci}) \leq 1 \) for \( NSTX \)

A roughly linear scaling with \( B_{\text{tor}} \) is seen for GAE frequency in \textbf{NSTX-U} [Fredrickson,NF18]
HYM simulations for DIII-D

Two basic cases are considered:

1. NSTX-similarity experiments on DIII-D from [W. Heidbrink et al, NF 2006]
   \[ B_{tor} = 0.6T, \quad R_0 = 1.63m, \quad a = 0.56m, \quad I = 0.6MA, \quad q_0 = 1.2, \quad q_{max} = 4.5, \quad \beta_{tot} \sim 9\% \]
   Beam parameters: \( E = 80\text{keV}, \quad V_0/V_A = 1.5, \quad n_b/n_e \sim 4\%, \quad \beta_{beam} \sim 3\% \)
   Observed mode parameters: \( f = 2.5\text{MHz}, \quad f_{ci} = 4.5\text{MHz} \)

   \[ B_{tor} = 1.24T, \quad R_0 = 1.72m, \quad I = 0.62MA, \quad \beta_{tot} \sim 2\% \]
   Beam parameters: \( E = 78\text{keV}, \quad V_0/V_A = 0.8, \quad n_b/n_e \sim 3\%, \quad \beta_{beam} \sim 0.5\% \)
   Observed mode parameters: \( f = 5.5\text{MHz}, \quad f_{ci} = 9.5\text{MHz} \)
Sub-cyclotron frequency Alfven Eigenmodes were observed in low toroidal field experiments in DIII-D

- High-frequency AEs were observed in DIII-D in low toroidal field discharges (NSTX similarity experiments) mostly when $v_b \gtrsim v_A$.
- These modes are counter-propagating and driven unstable by Doppler shifted cyclotron resonance with beam ions.
- Mode polarization was not measured directly, but large $dB\varphi$ was observed near the edge.
- They were identified as compressional Alfven eigenmodes (CAE) based on high frequency ($f \sim 0.6 f_{ci}$) and comparison with previous theoretical instability conditions.
- For comparison:
  
  $\text{NSTX} \rightarrow f_{GAE} \sim 0.1-0.3 \ f_{ci} \ ; \ f_{CAE} \sim 0.3-0.5 \ f_{ci}$
  
  $\text{NSTX-U} \rightarrow f_{GAE} \sim 0.4 \ f_{ci}$

Time evolution of the magnetic signal and spectra in a discharge with periodic injection of 80 keV left beams; $B_T = 0.6 \ T$; $f_{ci} = 4.5\text{Mhz}$. [Heidbrink, NF2006]
Early CAE/GAE theory predicted instability for large values of $k\perp \rho_b$

- Experimental estimates got $k\perp \rho_b \leq 1$ (based on CAE dispersion ie from observed $\omega/v_A$) and local B value.
- Re-scaling for B on axis gives even lower values: $k\perp \rho_b \leq 0.6 k\perp v_A/\omega$.
- Previously theory predicted the CAE instability for: $1 < k\perp \rho_b < 2$
- For Global Alfven eigenmode (GAE) instability: $2 < k\perp \rho_b < 4$
- New GAE/CAE theory [Belova,2019, Lestz, 2020] predicts stronger instability for small $k\perp/k\parallel$ ($k\perp \rho_b \ll 1$), and GAEs more unstable than CAEs.
Case 1: NSTX-similarity experiments on DIII-D

HYM parameters: \( V_0/V_A = 1.5, \ n_b/n_e = 4\%, \lambda_0 = 0.75, \Delta \lambda = 0.2 \)

Simulation results:
- Unstable modes have shear Alfven polarization with \( \delta B_\parallel < < \delta B_\perp \) and are counter-rotating GAEs
- \(|n| = 12-18\) with \( m \sim 1-5\),
- frequencies \( \omega/\omega_{ci} = 0.5-0.7\), and growth rates \( \gamma/\omega_{ci} = 0.003 - 0.0075\).
- Can estimate unstable \(|n|\) from \( n \approx R_0 k_\parallel \) and SAW dispersion to get: \( n \approx R_0 \omega/V_A \sim 15 \).

Toroidal mode numbers were not measured in experiments but estimated as \( n= -O(10)\).
For #120196 shot \( n= -16+/ - 5\) was inferred from correlation with changes in Mirnov signal [W. Heidbrink et al, NF 2006]
Modes have shear polarization in the core (GAEs)

- Simulations show unstable counter-propagating GAEs with peak values of $\delta B_\parallel \sim 0.05 \delta B_\perp$
- Mode is located near the magnetic axis $R=1.6\text{m}$; $\delta B_\parallel$ - radial profile is wider.
- $k_\perp \rho_b \sim 1$ for $m=2-3$, $v_\parallel = 0.7v$, and $v \sim 0.8v_0$
Compressional perturbations dominate at the edge

Time evolution of different $\delta B$ components for $n = -15$ counter-GAE at two radial locations away from the axis ($R_0=1.61m$): $R=1.96m$ and $R=2.25m$.

$\delta B_{||}$ has much wider radial profile compared to $\delta B_{\perp}$ => compressional perturbations dominate at the edge where: $\delta B_{||} \sim 2|\delta B_{\perp}|$. 
Large number of sideband resonances can be seen for each unstable GAE

HYM: fast-ion energy vs pitch distribution from n=-16 GAE simulations; resonant line is shown for \( v_\parallel = 0.8v_A \); colour dots show resonant particles.

Location of resonant particles in phase space: \( \lambda = \mu B_\phi / \varepsilon \) vs \( p_\phi \). Particle color corresponds to different energies: from \( E=0 \) (purple) to \( E=80 \) keV (red).
Density scan allows to estimate damping for GAE

- Estimated stability threshold is $n_b/n_0 \approx 0.02$.
- Dependence is not linear in contrast with NSTX and NSTX-U simulations.
- Assuming that $\gamma = Cn_b - \gamma_d$, the damping rate (continuum) can be calculated.
- For experimental parameters ($n_b/n_0 \approx 0.04$): $\gamma_d \approx 0.5 \gamma_{\text{drive}}$, where $\gamma_{\text{drive}}/\omega_{ci} \approx 0.016$.

Growth rate and frequency of $n= -16$ GAE vs beam density.
Case 2: Higher $B_{\text{tor}}$ experiments on DIII-D

$B_{\text{tor}} = 1.24T$, $R_0 = 1.72m$, $I= 0.62MA$, $q_0=0.94$, $q_{\text{max}}=6$, $\beta_{\text{tot}} \sim 2\%$

Beam parameters: $E= 75-80keV$, $V_0/V_A = 0.8$, $n_b/n_e = 3\%$, $\beta_{\text{beam}} \sim 0.5\%$

Observed mode parameters: $f= 5.5 \text{ MHz}$, $f_{ci} = 9.5 \text{ MHz}$

**HYM parameters:** $V_0/V_A = 0.8-0.9$, $n_b/n_e = 3-6\%$, $\lambda_0 = 0.65-0.75$, $\Delta \lambda = 0.2$

- Unstable modes have shear Alfven polarization with $\delta B_{\parallel} \ll \delta B_{\perp}$ and are counter-rotating – GAEs
- $|n|= 22-24$ with $m \sim 3-4$, 
- Frequencies $\omega/\omega_{ci} = 0.6-0.75$, and growth rates $\gamma/\omega_{ci} = 0.001 - 0.003$. 
- SAW estimate for $n$: $|n| = R_0 \omega/V_A \sim 19 - 22$, 

DIII-D estimated toroidal mode numbers was $n \approx 28$, from 2-fluid SAW dispersion relation [S.Tang, PRL 2021].

**Growth rates and frequencies of unstable counter-GAEs from HYM simulations for $V_0/V_A = 0.8$, $n_b = 6\%$, $\lambda_0 = 0.75$.**
HYM simulations demonstrate that unstable modes in DIII-D have SAW polarization (GAEs)

- Simulations show unstable counter-propagating GAEs with $\delta B_\perp > 10 \delta B_\parallel$, but $\delta B_\parallel$ has wider radial profile.
- High toroidal mode numbers $|n| > 20$; $\omega/\omega_{ci} \sim 0.6-0.7$; $k_\perp \rho_B \sim 0.5$
- Located near the magnetic axis.
Large number of sideband resonances can be seen for each unstable GAE

Location of resonant particles in phase space: \( \lambda = \mu B_0 / \epsilon \) vs \( p_\varphi \). Particle color corresponds to different energies: from \( E=0 \) (purple) to \( E=80 \) keV (red).

HYM fast-ion energy vs pitch distribution from \( n=-22 \) GAE simulations; resonant line is shown for \( v_{||}=0.4v_A \); colour dots show resonant particles.
Two groups of resonant particles: driving ($\lambda<\lambda_0$) or damping ($\lambda>\lambda_0$)

Energy exchange rate between the beam ions and the mode $\Delta K = \int (\delta j_b \cdot \delta E) d^3x = \sum (\nu_m \cdot \delta E)w_m$.

Lower energy particles are driving, and particles from the tail are stabilizing GAE.

Scatter plot of resonant particles from linear phase of $n=-22$ GAE simulations. Time-averaged values of $\langle \nu \cdot \delta E \rangle w$ [a.u.] of resonant particles. Particle color corresponds to different energies: from $E=35$keV (green) to $E=80$keV (red).
HYM simulations for DIII-D agree with analytic predictions

(a) contour plot of GAE growth rate and (b) plot of $\gamma(k_{\perp}p_i)$ for fixed frequencies; blue-green contours correspond to negative values, and orange-red to positive; $\gamma/\omega_{ci}$ values are between -0.08:0.011.

DIII-D beam parameters: $V_0/V_A=0.9$, $n_b/n_e=0.05$, $\lambda_0=0.7$, $\Delta\lambda=0.2$.

- Frequency of most unstable modes: $\omega/\omega_{ci} \sim 0.6$,
- GAE linear growth rate is largest for small values of $k_{\perp}$

$\gamma \approx \frac{\pi n_b \omega_{ci}^2}{n_i^2 2\omega_{ci} v_{\text{res}}^3} A \int_0^{\lambda_m} \frac{d\lambda}{(1-\lambda)^2} \frac{1}{\partial^2 \gamma/\gamma_{\text{res}}}$,

$v_{\text{res}} = (\omega_{ci} - \omega)/|k_{\parallel}|$, \quad $\lambda_m = 1 - v_{\text{res}}^2/v_0^2$, \quad $v_\theta$ – injection velocity, \quad $f = A \exp[-(2-\lambda_0)^2/\Delta\lambda^2] / (v^3+v_{\text{res}}^3)$.

- Resonant beam ions drive instability provided: $\partial f/\partial \lambda > 0$, i.e. when $\lambda < \lambda_0$ and stabilizing otherwise.
- Most unstable modes have $k_{\perp}p_i < 1$, and are in the range [1,2]:

$\omega/\omega_{ci} < 0$

$\omega/\omega_{ci} > 0$

$\gamma < 0$

$\gamma > 0$

$\gamma_{\text{res}} = (\omega_{ci} - \omega)/|k_{\parallel}|$  

$\lambda_m = 1 - v_{\text{res}}^2/v_0^2$, \quad $v_\theta$ – injection velocity, \quad $f = A \exp[-(2-\lambda_0)^2/\Delta\lambda^2] / (v^3+v_{\text{res}}^3)$.

Dependence on frequency and \( k_\perp \rho_i \) for counter-GAEs

\[
\gamma \approx \frac{n_b \omega_{ci}^2}{n_i 2\omega} v_{\|\text{res}}^2 A \int_0^{\lambda_m} \frac{d\lambda}{(1-\lambda)^2} \lambda \frac{\partial f}{\partial \lambda} J_1^2 v_{\|\text{res}} v_{\perp} \left( \frac{k_\perp}{\rho_i} \right),
\]

\[
G(\lambda) = \frac{\lambda - v_{\perp}^2/\lambda^2}{\lambda_m = 1 - v_{\|\text{res}}^2/v_0^2}, \quad v_{\perp\|\text{res}} = (\omega_{ci} - \omega)\|k||
\]

\[
\xi = k_\perp v_{\perp}/\omega_{ci}
\]

\[
f = A \exp\left[ -\left(\lambda - \lambda_0\right)^2/\Delta \lambda^2 \right] / (v^3 + v_3^3)
\]

The sign of the integrand is determined by sign of \( \partial f/\partial \lambda = 2(\lambda_0 - \lambda)f/\Delta \lambda^2 \)

→ particles with \( \lambda < \lambda_0 \) (small \( v_{\perp} \)) are destabilizing,
→ particles with \( \lambda > \lambda_0 \) (large \( v_{\perp} \)) are stabilizing.

1. For \( \omega < \omega_{ci} \), \( v_{\|\text{res}} \approx v_0 \) and \( \lambda_m < 1 \)
   • \( \lambda_m \leq \lambda_0 \) – sufficient condition for instability (any \( k_\perp \)) and gives an approximate range of unstable frequencies:
     \( \omega/\omega_{ci} \leq (1 + v_0/v_A \sqrt{[1-\lambda_0]})^{-1} \)
   • most unstable modes have \( k_\perp \rho_i \ll 1 \) with \( (J_1/\xi)^2 \approx 1/4 \)

2. High-frequency limit \( \omega \approx \omega_{ci} \), \( v_{\|\text{res}} \ll v_0 \) and \( \lambda_m \approx 1 \)
   • for small \( k_\perp \rho_i \) → \( (J_1/\xi)^2 = 1/4 \) and \( \gamma \) is negative
   • for \( k_\perp \rho_i \approx 2 \), Bessel factor reduces stabilizing effect of large \( v_{\perp} \) \( (\lambda > \lambda_0) \) particles.

This clarifies the role of FLR effects: large \( k_\perp \rho_i \) reduces the stabilizing effect of particles with \( \lambda > \lambda_0 \)

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This clarifies the role of FLR effects: large \( k_\perp \rho_i \) reduces the stabilizing effect of particles with \( \lambda > \lambda_0 \)
Counter-CAE are predicted to be less unstable than GAEs.

Contour plots of growth rate and plots of $\gamma$ vs $k_{\perp}\rho_i$ for wide pitch parameter distributions; blue-green contours correspond to negative values, and orange-red to positive.

(a) $\Delta \lambda_{GAE}=0.3$, GAE

(b) $\Delta \lambda_{CAE}=0.3$, CAE

Contour plots of growth rate and plots of $\gamma$ vs $k_{\perp}\rho_i$ for wide pitch parameter distributions; blue-green contours correspond to negative values, and orange-red to positive.

(a) GAE: $\Delta \lambda_{GAE}=0.3$, $\gamma/\omega_{ci}$ values are between $-0.22 \pm 0.036$;

(b) CAE: $\Delta \lambda_{CAE}=0.3$, same range of $\gamma/\omega_{ci}$ values as in (a).

\[ \gamma \approx \pi \frac{n_b \omega^2_{ci}}{2 \omega_i v_{res} A} \int_0^{\lambda_m} \left( \frac{d \lambda}{\partial \lambda} \right)^3 \frac{\partial f}{\partial \lambda} \left( \frac{j_0^2 - j_2^2}{4} \right) \mathrm{d} \lambda, \]

where $v_{||}=v_{res}$.

- Same (sufficient) instability condition: $\lambda_m < \lambda_0$ as GAEs, and same $\gamma$ for $k_{\perp}=0$ [1,2], but $\omega = kv_A$.

- Most unstable modes have $k_{\perp}\rho_i < 1$, and are in the range:

  \[ (1 + \alpha v_0/v_A)^{-1} < \omega/\omega_{ci} < (1 + \alpha v_0/v_A\sqrt{1-\lambda_0})^{-1} \]

  where $\alpha = |k_{||}/k|$.

- Counter-CAEs have much smaller growth rates than GAEs for $k_{\perp}/k_{||} \gtrsim 1$.

- Two-fluid effects / coupling to SAW reduce growth rate of CAEs [2].

Finite frequency corrections for GAEs $\omega/\omega_{ci} \sim 1$

$$\omega^2 = k_{||}^2 v_A^2 f^2,$$

where

$$f^2 = \frac{1}{2} \left[ \left( \frac{k^2}{k_{||}^2} (1 + \chi_{||}^2) + 1 \right) - \sqrt{ \left( \frac{k^2}{k_{||}^2} (1 + \chi_{||}^2) + 1 \right)^2 - 4 \frac{k^2}{k_{||}^2} } \right],$$

and $\chi_{||} = k_{||} v_A / \omega_{ci}$

Limit $k_{\perp} = 0$:

$$\omega = k_{||} v_A \left( \sqrt{1 + \left( \frac{k_{||} v_A}{2 \omega_{ci}} \right)^2} - \frac{k_{||} v_A}{2 \omega_{ci}} \right)$$

Limit $k_{\perp} \gg k_{||}$:

$$\omega = k_{||} v_A / \sqrt{1 + \left( \frac{k_{||} v_A}{\omega_{ci}} \right)^2}$$

- $\lambda_m = 1 - v_{||res}^2 / v_0^2 \leq \lambda_0$ – sufficient condition for instability ($\approx$ peak growth rate) valid for large frequencies ($\omega/\omega_{ci} \sim 1$) but correct 2-fluid GAE dispersion should be used.
- One-fluid MHD dispersion leads to overestimated most unstable frequency and underestimated $k_{||}$ (n).
Most unstable frequency and $k_{\parallel}$ range for $\omega/\omega_{ci} \sim 1$

MHD conditions: 
\[
\left(1 + \frac{v_0}{v_A}\right)^{-1} < \frac{\omega}{\omega_{ci}} \leq \left(1 + \frac{v_0}{v_A} \sqrt{1 - \lambda_0}\right)^{-1},
\]
corresponding to $v_{\parallel, res} < v_0$ and $\lambda_m \leq \lambda_0$ respectively, modified for $\omega/\omega_{ci} \sim 1$. Due to MHD description of thermal plasma, the HYM code overestimates most unstable frequencies and underestimates most unstable $|n|$. 

Numerical solution for two conditions using 2-fluid SAW dispersion:
1. $v_{\parallel, res} \leq v_0$ (existence of resonance)
2. $\lambda_m = 1 - v_{\parallel, res}^2/v_0^2 \leq \lambda_0$ (max $\gamma$)
HYM simulations demonstrate that high-frequency modes ($\omega/\omega_{ci} \sim 0.6$) previously misidentified in DIII-D as compressional (CAEs), have shear polarization $\delta B \approx \delta B_\perp$ (GAEs).

Simulations reproduce experimentally observed frequencies and estimated toroidal mode numbers for DIII-D experiments.

A simple analytical theory based on local dispersion relation is very successful in predicting the counter-GAE instabilities.

New analytic theory explains range of most unstable modes, and GAE frequency scaling across different devices (NSTX, NSTX-U, DIII-D).

Counter-GAEs can be unstable in ITER ($v_{beam}/v_A \leq 1$) with $\omega/\omega_{ci} \sim 0.5-0.7$.

Future work:

Need to include 2-fluid (Hall) effects in thermal plasma description to account for finite frequency effects $\sim O(\omega/\omega_{ci})$. At present, HYM overestimates unstable frequencies, and underestimates toroidal mode numbers.