First-principle formulation of resonance broadened quasilinear theory near an instability threshold

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“The collisional resonance function in discrete-resonance quasilinear plasma systems”, (arXiv:1906.01780)
Critical gradient behavior suggests that quasilinear modeling is appropriate

DIII-D critical gradient experiments
Stiff, stochastic fast ion transport gives credence in using a quasilinear approach

- Fully nonlinear modeling of fast ion interaction with Alfvénic modes in a realistic tokamak is numerically expensive
- Reduced (but still realistic) modeling can be exploited if linear mode properties do not change faster than the equilibrium, e.g.,
  - eigenstructure
  - resonance condition
- Simulations need to cope with the simultaneous excitation of multiple unstable Alfvénic instabilities
- Reduced models need to be strongly verified
Early development of broadening quasilinear theory

Let us consider canonical variables of actions $J$ and angles $\varphi$

In a tokamak, $J$ is a combination of ($E$, $P_\varphi$, $\mu$)

\[
\dot{\varphi} = \frac{\partial H_0}{\partial J} / \partial J = \Omega(J)
\]

The line broadening model $(\delta(\Omega - \omega) \rightarrow \mathcal{F}(\Omega - \omega))$

\[
\frac{\partial f}{\partial t} = \frac{\pi}{2} \frac{\partial}{\partial \Omega} \omega_b^4 \mathcal{F} \frac{\partial f}{\partial \Omega} + \nu_{\text{eff}}^3 \frac{\partial^2 f}{\partial \Omega^2} (f - f_0)
\]

\[
\frac{d}{dt} \omega_b^4 = 2(\gamma - \gamma_d)\omega_b^4
\]

\[
\gamma = \frac{\pi}{4} \int d\Omega \mathcal{F} \frac{\partial f}{\partial \Omega}
\]

- $\mathcal{F}$ is an arbitrary resonance function
- $\omega_b$ is the trapping (bounce) frequency at the elliptic point (proportional to square root of mode amplitude)


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Determining the parametric dependencies of the broadening from single mode saturation levels

We use analytic results for determining $a$ and $b$: $\Delta \Omega = a \omega_b + b \nu_{eff}$

Limit near marginal stability\(^3\)

$\omega_b = 1.18 \nu_{eff} \left( \frac{\gamma_L - \gamma_d}{\gamma_L} \right)^{1/4}$

$\rightarrow b = 3.1$

Limit far from marginal stability\(^4\)

$\omega_b = 1.2 \nu_{eff} \left( \frac{\gamma_L - \gamma_d}{\gamma_d} \right)^{1/3}$

$\rightarrow a = 2.7$

Resonance-broadened quasilinear formalism can cope with both situations of isolated and overlapping modes


The overlapping of resonances lead to losses due to global diffusion

- Designed to address both regimes of isolated and overlapping resonances
  - the fast ion distribution function relaxes while self-consistently evolving the amplitude of modes

First-principle analytical determination of the collisional resonance broadening – part I

Start with the kinetic equation:

\[
\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \varphi} + \text{Re} \left( \omega_b^2 e^{i\varphi} \right) \frac{\partial f}{\partial \Omega} = C [f, F_0]
\]

\[
\nu_K (F_0 - f)
\]

\[
\nu_{\text{scatt}} \frac{d^2}{d\Omega^2} (f - F_0)
\]

Periodicity over the canonical angle allow the distribution to be written as a Fourier series:

\[
f (\varphi, \Omega, t) = F_0 (\Omega) + f_0 (\Omega, t) + \sum_{n=1}^{\infty} \left( f_n (\Omega, t) e^{in\varphi} + \text{c.c.} \right)
\]

Near marginal stability, a perturbation theory can be developed in orders of \( \omega_b^2 / \nu_{K, \text{scatt}} \), which leads to the ordering \( |F'_0| \gg |f''_1| \gg |f''_0|, |f''_2| \). When memory effects are weak, i.e., \( \nu_{K, \text{scatt}} / (\gamma_{L,0} - \gamma_d) \gg 1 \),

\[
f_1 = \frac{\omega_b^2 F'_0}{2 (i\Omega + \nu_K)} \quad \frac{\partial f_0}{\partial t} + \frac{1}{2} \left( \omega_b^2 |f'_1|^* + \omega_b^2 f'_1 \right) = -\nu_K f_0
\]

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First-principle analytical determination of the collisional resonance broadening – part II

When decoherence is strong, the distribution function has no angle dependence:

\[ f(\Omega, t) \equiv F_0(\Omega) + f_0(\Omega, t) \]

In the limit \( \nu_{K, \text{scatt}} / (\gamma_L, 0 - \gamma_d) \gg 1 \), the distribution satisfies a diffusion equation:

\[ \frac{\partial f(\Omega, t)}{\partial t} - \frac{\pi}{2} \frac{\partial}{\partial \Omega} \left[ |\omega_b|^2 |\mathcal{R}(\Omega) \frac{\partial f(\Omega, t)}{\partial \Omega} \right] = C[f, F_0] \]

With the spontaneously emerged collisional resonance functions (both satisfy \( \int_{-\infty}^{\infty} F(\Omega)d\Omega = 1 \)):

\[ \mathcal{R}_K(\Omega) = \frac{1}{\pi \nu_K (1 + \Omega^2 / \nu_K^2)} \quad \mathcal{R}_{\text{scatt}}(\Omega) = \frac{1}{\pi \nu_{\text{scatt}}} \int_0^\infty ds \cos \left( \frac{\Omega s}{\nu_{\text{scatt}}} \right) e^{-s^3/3} \]

Self-consistent formulation of collisional quasilinear transport theory

\[ \frac{\partial f(\Omega, t)}{\partial t} - \frac{\pi}{2} \frac{\partial}{\partial \Omega} \left[ |\omega_b|^2 \mathcal{R}(\Omega) \frac{\partial f(\Omega, t)}{\partial \Omega} \right] = C[f, F_0] \]

\[ \gamma_L(t) = \frac{\pi}{4} \int_{-\infty}^{\infty} d\Omega \mathcal{R} \frac{\partial f(\Omega, t)}{\partial \Omega} \quad d\left| \omega_b^2 \right|^2 / dt = 2(\gamma_L(t) - \gamma_d) \left| \omega_b^2 \right|^2 \]

- A QL theory naturally emerges when considering kinetic theory near threshold when collisions occur at a time scale faster than the phase mixing time scale.

- The QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels calculated from full kinetic theory near marginality, with a rather complex time-delayed integro-differential equation (Berk, Breizman and Pekker, *Phys. Rev. Lett.* 1996) \[ |\omega_{b, sat}| = 8^{1/4} (1 - \gamma_d/\gamma_{L,0})^{1/4} \nu_K \]
Summary

• A systematic QL theory has been derived from first principles near an instability threshold.
• Indicates that QL theory is applicable to a single discrete resonance (with no overlap), provided that stochasticity is large enough
• Collisional resonance broadening functions emerge spontaneously
• Major arbitrariness of collisional QL theory (the shape of the resonance functions) has now been removed
• The quasilinear system (with the calculated broadening functions) systematically recovers the mode saturation levels for near-threshold plasmas previously calculated from nonlinear kinetic theory
• Resonance functions are being implemented into the Resonance Broadening Quasilinear (RBQ) code

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Verification: analytical collisional mode evolution near threshold

- Near marginal stability, the wave amplitude evolution is governed by [Berk, Breizman and Pekker, PRL 1996]

\[
\frac{dA(t)}{dt} = A(t) - \frac{1}{2} \int d\Gamma \mathcal{H} \left\{ \int_0^{t/2} dz z^2 A(t - z) \times \int_0^{t-2z} dy e^{-\hat{v}_{eff}^3 z^2 (2z/3 + y)} A(t - z - y) A^*(t - 2z - y) \right\}
\]

- An approximate analytical solution is found when \( \hat{v}_{eff} \gg 1 \): [Duarte & Gorelenkov, NF 2019]

\[
A(t) = \frac{A(0)e^t}{\sqrt{1 - gA^2(0) (1 - e^{2t})}}
\]

\[
g \equiv \int d\Gamma \mathcal{H} \frac{\Gamma(1/3)}{6\hat{v}_{eff}^2} \left( \frac{3}{2} \right)^{1/3}
\]

is a resonance-averaged collisional contribution evaluated by NOVA-K

Amplitude A vs time t for the full cubic equation (green) and the analytical solution (black)

Vinicius Duarte, “First-principle formulation of resonance broadened quasilinear theory,” [Duarte & Gorelenkov, NF 2019]
Resonance-broadened quasilinear (RBQ) diffusion model

Formulation in action and angle variables\textsuperscript{2,3}
- Diffusion equation:
  \[
  \frac{df}{dt} = \frac{\partial}{\partial I} \left( \sum_{n_k,p,m,m'} D(I; t) \right) \frac{df}{\partial I} + \left( \frac{\partial \Omega_1}{\partial I} \right)^{-2} \nu_{\text{scat,1}}^3 \frac{\partial^2 (f - f_0)}{\partial I^2}
  \]
- Mode amplitude evolution:
  \[
  \frac{dC_n^2(t)}{dt} = 2 (\gamma_{L,n} - \gamma_{d,n}) C_n^2(t)
  \]

Broadening is the platform that allows for momentum and energy exchange between particles and waves:

\[
\Delta \Omega = a \omega_b + b \nu_{\text{eff}}
\]

\textsuperscript{1}Berk, Breizman, Fitzpatrick and Wong, NF 1995.
\textsuperscript{2}Kaufman PoF, JPP 1972 (no broadening due to growth rate!).
\textsuperscript{3}Gorelenkov, Duarte, Podestà and Berk, NF 2018.
Broadening is adjusted to replicate analytical predictions for the mode saturation amplitude of single modes

Definitions: initial linear growth rate $\gamma_L$, mode damping rate $\gamma_d$ and trapping (bounce) frequency $\omega_b$ (proportional to square root of mode amplitude)

### Collisionless case

- Undamped case

$$\omega_b \approx 3.2\gamma_L$$

### Collisional cases

- Close to marginal stability: $\nu_{\text{eff}} \gg \omega_b$

$$\omega_b = 1.18\nu_{\text{eff}} \left(\frac{\gamma_L - \gamma_d}{\gamma_d}\right)^{1/4}$$

- Far from marginal stability: $\omega_b \gg \nu_{\text{eff}}$

$$\omega_b = 1.2\nu_{\text{eff}} \left(\frac{\gamma_L - \gamma_d}{\gamma_d}\right)^{1/3}$$
same form of the function calculated by Dupree [T. H. Dupree, Phys. Fluids 9, 1773 (1966)] in a different context, namely in the study of strong turbulence theory, where a dense spectrum of fluctuations diffuse particles away from their free-streaming trajectories. In that case, the cubic term in the argument of the exponential is proportional to a collisionless diffusion coefficient.

- the reduction of reversible equations of motion into a diffusive system of equations that governs the resonant particle dynamics without detailed tracking of the ballistic motion

- The collisional broadening of resonance lines is a universal phenomenon in physics (e.g., atoms emission/absorption spectral profile in atomic physics)