# First-principle formulation of resonance broadened quasilinear theory near an instability threshold

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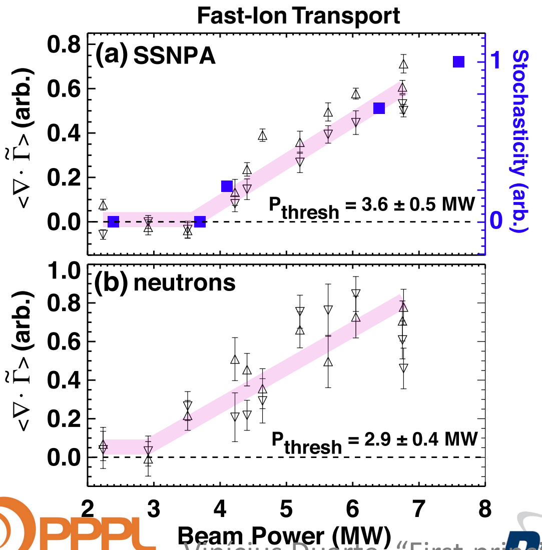
*"The collisional resonance function in discrete-resonance quasilinear plasma systems"*, (arXiv:1906.01780)

# Critical gradient behavior suggests that quasilinear modeling is appropriate

## DIII-D critical gradient experiments

Stiff, stochastic fast ion transport gives credence in using a quasilinear approach




C. Collins et al, PRL 2016

- Fully nonlinear modeling of fast ion interaction with Alfvénic modes in a realistic tokamak is numerically expensive
- Reduced (but still realistic) modeling can be exploited if linear mode properties do not change faster than the equilibrium, e. g.,
  - eigenstructure
  - resonance condition
- Simulations need to cope with the simultaneous excitation of multiple unstable Alfvénic instabilities
- Reduced models need to be strongly verified

# Early development of broadening quasilinear theory

Let us consider canonical variables of actions $J$ and angles $\varphi$

$$\dot{\varphi} = \partial H_0(J)/\partial J \equiv \Omega(J)$$

In a tokamak, $J$ is a combination of $(\mathcal{E}, P_\varphi, \mu)$

**The line broadening model** $(\delta(\Omega-\omega) \rightarrow \mathcal{F}(\Omega-\omega))$:

$$\frac{\partial f}{\partial t} = \frac{\pi}{2}\frac{\partial}{\partial \Omega}\omega_b^4 \mathcal{F}\frac{\partial f}{\partial \Omega} + \nu_{\mathrm{eff}}^3 \frac{\partial^2}{\partial \Omega^2}(f - f_0) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\omega_b^4 = 2(\gamma - \gamma_d)\omega_b^4 \qquad \gamma = \frac{\pi}{4}\int \mathrm{d}\Omega \mathcal{F}\frac{\partial f}{\partial \Omega}$$

- $\mathcal{F}$ is an arbitrary resonance function
- $\omega_b$ is the trapping (bounce) frequency at the elliptic point (proportional to square root of mode amplitude)

H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

# Determining the parametric dependencies of the broadening from single mode saturation levels

We use analytic results for determining $a$ and $b$: $\Delta\Omega = a\omega_b + b\nu_{eff}$

Limit near marginal stability[3]
→ $b = 3.1$

$$\omega_b = 1.18\nu_{eff}\left(\frac{\gamma_{L0}-\gamma_d}{\gamma_{L0}}\right)^{1/4}$$

Limit far from marginal stability[4]
→ $a = 2.7$

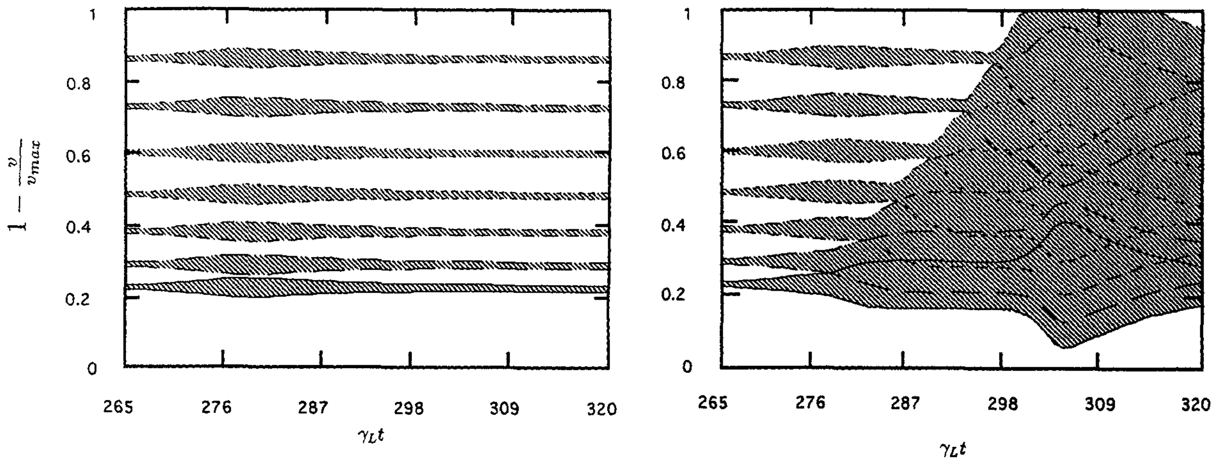
$$\omega_b = 1.2\nu_{eff}\left(\frac{\gamma_{L0}-\gamma_d}{\gamma_d}\right)^{1/3}$$

Resonance-broadened quasilinear formalism can cope with both situations of isolated and overlapping modes

[3] H. L. Berk et al. Plasma Phys. Rep, 23(9), 1997
[4] H. L. Berk and B. N. Breizman. Phys. Fluids B, 2(9), 1990

# The overlapping of resonances lead to losses due to global diffusion

- Designed to address both regimes of isolated and overlapping resonances
  - the fast ion distribution function relaxes while self-consistently evolving the amplitude of modes




H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

# First-principle analytical determination of the collisional resonance broadening – part I

Start with the kinetic equation: $\frac{\partial f}{\partial t}+\Omega\frac{\partial f}{\partial \varphi}+Re\left(\omega_b^2 e^{i\varphi}\right)\frac{\partial f}{\partial \Omega}=C\left[f,F_0\right]$

$\nu_K\left(F_0-f\right)$

$\nu_{scatt}^3\partial^2\left(f-F_0\right)/\partial\Omega^2$

Periodicity over the canonical angle allow the distribution to be written as a Fourier series:

$$f\left(\varphi,\Omega,t\right)=F_0\left(\Omega\right)+f_0\left(\Omega,t\right)+\sum_{n=1}^{\infty}\left(f_n\left(\Omega,t\right)e^{in\varphi}+c.c.\right)$$

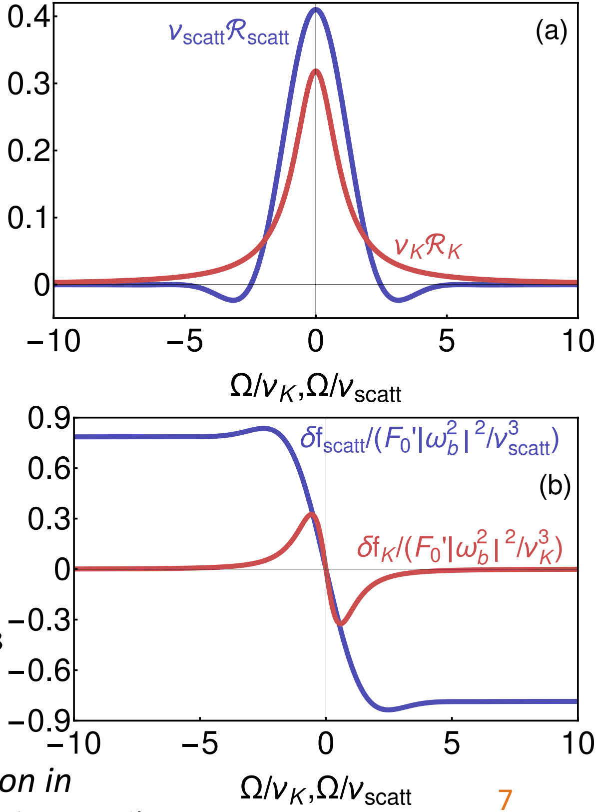
Near marginal stability, a perturbation theory can be developed in orders of $\omega_b^2/\nu_{K,scatt}^2$ which leads to the ordering $\left|F_0'\right|\gg\left|f_1'^{(1)}\right|\gg\left|f_0'^{(2)}\right|,\left|f_2'^{(2)}\right|$. When memory effects are weak, i.e., $\nu_{K,scatt}/\left(\gamma_{L,0}-\gamma_d\right)\gg 1$,

$$f_1=\frac{\omega_b^2 F_0'}{2\left(i\Omega+\nu_K\right)} \qquad \frac{\partial f_0}{\partial t}+\frac{1}{2}\left(\omega_b^2\left[f_1'\right]^*+\omega_b^{2*}f_1'\right)=-\nu_K f_0$$

# First-principle analytical determination of the collisional resonance broadening – part II

When decoherence is strong, the distribution function has no angle dependence:

$$f\left(\Omega,t\right)\equiv F_0\left(\Omega\right)+f_0\left(\Omega,t\right)$$




In the limit $\nu_{K,scatt}/\left(\gamma_{L,0}-\gamma_d\right)\gg 1$, the distribution satisfies a diffusion equation:

$$\frac{\partial f\left(\Omega,t\right)}{\partial t}-\frac{\pi}{2}\frac{\partial}{\partial\Omega}\left[\left|\omega_b^2\right|^2\mathcal{R}\left(\Omega\right)\frac{\partial f\left(\Omega,t\right)}{\partial\Omega}\right]=C\left[f,F_0\right]$$

With the spontaneously emerged collisional resonance functions (both satisfy $\int_{-\infty}^{\infty}\mathcal{F}(\Omega)d\Omega=1$):

$$\mathcal{R}_K(\Omega)=\frac{1}{\pi\nu_K\left(1+\Omega^2/\nu_K^2\right)}\qquad \mathcal{R}_{scatt}\left(\Omega\right)=\frac{1}{\pi\nu_{scatt}}\int_0^\infty ds\,\cos\left(\frac{\Omega s}{\nu_{scatt}}\right)e^{-s^3/3}$$

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*Duarte, Gorelenkov, White and Berk, "*The collisional resonance function in discrete-resonance quasilinear plasma systems*" arXiv:1906.01780v1 (submitted)

# Self-consistent formulation of collisional quasilinear transport theory

$$\frac{\partial f(\Omega,t)}{\partial t}-\frac{\pi}{2}\frac{\partial}{\partial\Omega}\left[\left|\omega_b^2\right|^2\mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial\Omega}\right]=C\left[f,F_0\right]$$

$$\gamma_L(t)=\frac{\pi}{4}\int_{-\infty}^{\infty}d\Omega\mathcal{R}\frac{\partial f(\Omega,t)}{\partial\Omega}\qquad d\left|\omega_b^2\right|^2/dt=2\left(\gamma_L(t)-\gamma_d\right)\left|\omega_b^2\right|^2$$

- A QL theory naturally emerges when considering kinetic theory near threshold when collisions occur at a time scale faster than the phase mixing time scale.

- The QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels calculated from full kinetic theory near marginality, with a rather complex time-delayed integro-differential equation (Berk, Breizman and Pekker, *Phys. Rev. Lett.* 1996) $\left|\omega_{b,sat}\right|=8^{1/4}\left(1-\gamma_d/\gamma_{L,0}\right)^{1/4}\nu_K$

# Summary

- A systematic QL theory has been derived from first principles near an instability threshold.
- Indicates that QL theory is applicable to a single discrete resonance (with no overlap), provided that stochasticity is large enough
- Collisional resonance broadening functions emerge spontaneously
- Major arbitrariness of collisional QL theory (the shape of the resonance functions) has now been removed
- The quasilinear system (with the calculated broadening functions) systematically recovers the mode saturation levels for near-threshold plasmas previously calculated from nonlinear kinetic theory
- Resonance functions are being implemented into the Resonance Broadening Quasilinear (RBQ) code

# Backup slides

# Verification: analytical collisional mode evolution near threshold

- Near marginal stability, the wave amplitude evolution is governed by [Berk, Breizman and Pekker, PRL 1996]

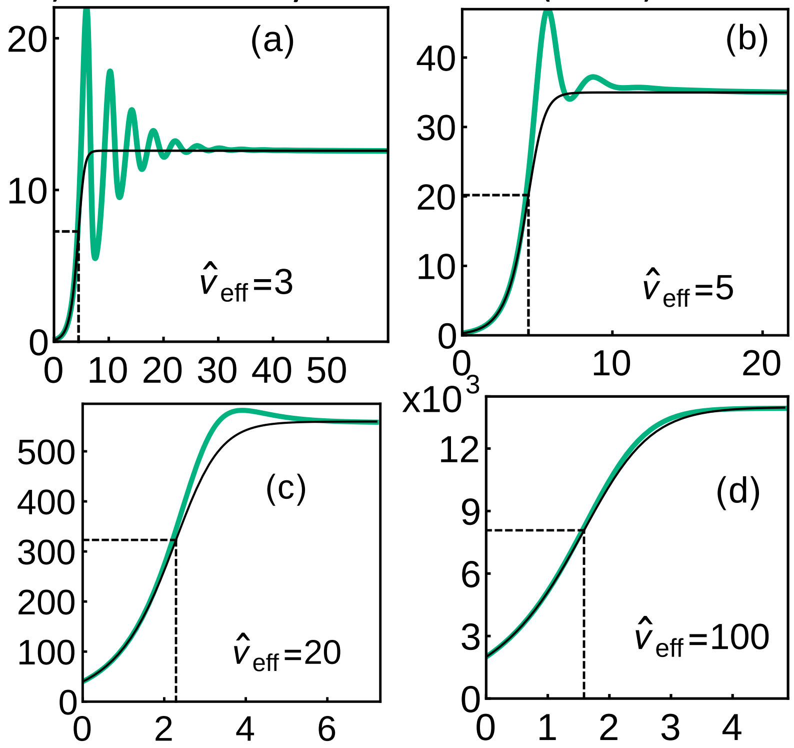
$$\frac{dA(t)}{dt} = A(t) - \frac{1}{2}\int d\Gamma \mathcal{H}\left\{\int_0^{t/2} dz z^2 A(t-z)\times \right.$$
$$\left. \times \int_0^{t-2z} dy e^{-\hat{\nu}_{eff}^3 z^2 (2z/3+y)} A(t-z-y) A^*(t-2z-y)\right\}$$

- An approximate analytical solution is found when $\hat{\nu}_{eff} \gg 1$: [Duarte & Gorelenkov, NF 2019]

$$A(t) = \frac{A(0)e^t}{\sqrt{1 - gA^2(0)\left(1-e^{2t}\right)}}$$

$g \equiv \int d\Gamma \mathcal{H} \frac{\Gamma(1/3)}{6\hat{\nu}_{eff}^4}\left(\frac{3}{2}\right)^{1/3}$ is a resonance-averaged collisional contribution evaluated by NOVA-K

Amplitude A vs time t for the full cubic equation (green) and the analytical solution (black)




[Duarte & Gorelenkov, NF 2019]

# Resonance-broadened quasilinear (RBQ) diffusion model

Formulation in action and angle variables[2,3]

- Diffusion equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I}\left(\sum_{n_k,p,m,m'} D(I;t)\right)\frac{\partial f}{\partial I} + \left(\left|\frac{\partial \Omega_{\mathbf{l}}}{\partial I}\right|_{I_r}\right)^{-2} \nu^3_{scatt,\mathbf{l}} \frac{\partial^2 (f - f_0)}{\partial I^2}$$

- Mode amplitude evolution:

$$\frac{dC_n^2(t)}{dt} = 2\left(\gamma_{L,n} - \gamma_{d,n}\right) C_n^2(t)$$

Broadened delta, a function of $\Delta\Omega$

$$D(I;t) = \pi C_k^2(t)\, \mathcal{E}^2 \frac{\mathcal{F}_{\mathbf{l}}(I - I_r)}{\left|\frac{\partial \Omega_{\mathbf{l}}}{\partial I}\right|} G^*_{m'p} G_{mp}$$

$$\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial \mathcal{E}} - n \frac{\partial}{\partial P_\varphi}$$

eigenstucture information

Physics-based determination of the window function is pending

Broadening is the platform that allows for momentum and energy exchange between particles and waves:

$$\boxed{\Delta\Omega = a\omega_b + b\nu_{eff}}$$

[1]Berk, Breizman, Fitzpatrick and Wong, NF 1995.
[2]Kaufman PoF, JPP 1972 (no broadening due to growth rate!).
[3]Gorelenkov, Duarte, Podestà and Berk, NF 2018.

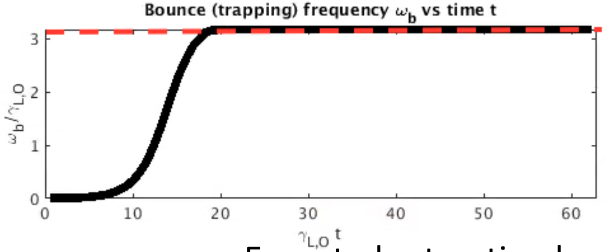
# Broadening is adjusted to replicate analytical predictions for the mode saturation amplitude of single modes

Definitions: initial linear growth rate $\gamma_L$, mode damping rate $\gamma_d$ and trapping (bounce) frequency $\omega_b$ (proportional to square root of mode amplitude)

## Collisionless case

- Undamped case

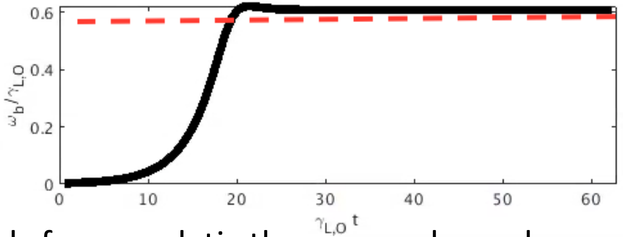
$$\omega_b \cong 3.2\gamma_L$$




## Collisional cases

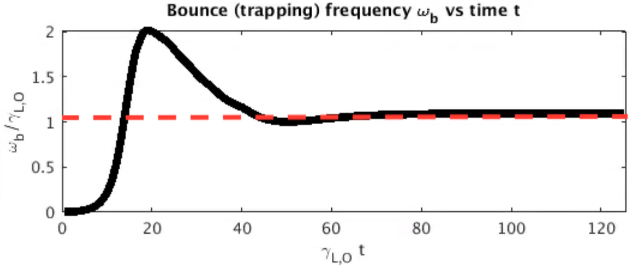
- Close to marginal stability: $\nu_{\text{eff}} \gg \omega_b$

$$\omega_b = 1.18\nu_{\text{eff}}\left(\frac{\gamma_L-\gamma_d}{\gamma_d}\right)^{1/4}$$




- Far from marginal stability: $\omega_b \gg \nu_{\text{eff}}$

$$\omega_b = 1.2\nu_{\text{eff}}\left(\frac{\gamma_L-\gamma_d}{\gamma_d}\right)^{1/3}$$




Expected saturation levels from analytic theory are shown by – – –

- same form of the function calculated by Dupree [T. H. Dupree, Phys. Fluids **9**, 1773 (1966)] in a different context, namely in the study of strong turbulence theory, where a dense spectrum of fluctuations diffuse particles away from their free-streaming trajectories. In that case, the cubic term in the argument of the exponential is proportional to a collisionless diffusion coefficient.
- the reduction of reversible equations of motion into a diffusive system of equations that governs the resonant particle dynamics without detailed tracking of the ballistic motion
- The collisional broadening of resonance lines is a universal phenomenon in physics (e.g., atoms emission/absorption spectral profile in atomic physics)