P2-16

Numerical analysis of two-fluid and finite Larmor radius effects on reduced MHD equilibrium with flow

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Equilibrium with flow in high-beta reduced MHD models

- Equilibrium with flow in fusion plasmas

  - In improved confinement modes of magnetically confined plasmas, equilibrium toroidal and poloidal flows play important roles like the suppression of instability and turbulent transport.

- Pressure anisotropy in plasma flow

  - Plasma flows driven by neutral beam injection indicate strong pressure anisotropy.

- Small scale effects in MHD equilibria

  - Equilibrium models with small scale effects may be suitable for modeling steady states of improved confinement modes that have steep plasma profiles and for initial states of multi-scale simulation.

  - Two-fluid equilibria with flow and pressure anisotropy was studied for the case of cold ions [Ito, Ramos and Nakajima, PoP **14**, 062502 (2007)].

  - However, finite ion Larmor radius effects may be relevant for high-temperature plasmas in magnetic confinement fusion devices.

$\rho_i = d_i\sqrt{\beta_i}$, ρ_i : ion Larmor radius, d_i :ion skin depth, $\beta_i \equiv p_i \big/ \left(B_0^2 / \mu_0 \right)$

- Equilibrium with flow in high-beta reduced MHD models

  - Fluid moments in collisionless, magnetized plasmas are simplified
  - Grad-Shafranov type equilibrium equations can be easily derived even in the presence of flow and several small scale effects.
  - Basic physics of flow and non-ideal effects can be investigated.
  - Reduced equilibrium models
    - Two-fluid MHD, FLR, poloidal Alfvenic flow
      [Ito, Ramos and Nakajima, PFR **3**, 034 (2008)]
    - MHD, poloidal sonic flow
      [Ito, Ramos and Nakajima, PFR **3**, 034 (2008)]
      [Ito and Nakajima, PPCF **51**, 035007 (2009)] – Analytic solution
    - Two-fluid MHD, FLR, poloidal sonic flow, isotropic pressure
      [Ito and Nakajima, AIP Conf. Proc. 1069, 121 (2008)]
    - Two-fluid MHD, FLR, poloidal sonic flow, anisotropic pressure
      [Ito and Nakajima, NF **51**, 123006 (2011)]
      [Ito and Nakajima, JPSJ 82, 064502 (2013)] – Analytic solution in the MHD limit

- Equilibrium equations in extended-MHD

- Fluid-moment equations for magnetized collisionless plasmas
[Ramos, PoP **12** 052102, 112301 (2005)]

- Electron inertia is neglected: $m_e \approx 0$

- Two-fluid equilibrium equations with ion FLR

$$\begin{aligned}(\lambda_H, \lambda_i) &= (0,0) \quad \Rightarrow \quad \text{Single-fluid MHD} \\ &= (1,0) \quad \Rightarrow \quad \text{Two-fluid (Hall) MHD} \\ &= (1,1) \quad \Rightarrow \quad \text{FLR two-fluid MHD}\end{aligned}$$

$$\nabla \cdot (n\mathbf{v}) = 0, \quad \nabla \times \mathbf{E} = 0, \quad \mu_0 \mathbf{j} = \nabla \times \mathbf{B}, \quad \mathbf{E} \equiv -\nabla \Phi,$$

$$m_i n \mathbf{v} \cdot \nabla \boldsymbol{v} = \mathbf{j} \times \mathbf{B} - \sum_{s=i,e} \left[\nabla p_{s\perp} + \mathbf{B} \cdot \nabla \left(\frac{p_{s\parallel} - p_{s\perp}}{B^2} \mathbf{B} \right) \right] - \lambda_i \nabla \cdot \Pi_i^{gv},$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{\lambda_H}{ne} \left\{ \mathbf{j} \times \mathbf{B} - \left[\nabla p_{e\perp} + \mathbf{B} \cdot \nabla \left(\frac{p_{e\parallel} - p_{e\perp}}{B^2} \mathbf{B} \right) \right] \right\},$$

$$\mathbf{v} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\perp}, \quad \mathbf{b} \equiv \mathbf{B} / B$$

- FLR effect (λ_i): ion gyroviscosity $\Pi_i^{gv} \sim \varepsilon^2 p$

- Two-fluid effects (λ_H): Hall current and electron pressure

- Equations for anisotropic ion and electron pressures

$\lambda_{i\|\|} = 0$	$\Rightarrow$	adiabatic ion pressure
$= 1$	$\Rightarrow$	ion pressure with parallel heat flux

$$\mathbf{v}\cdot\nabla p_{i\perp} + 2p_{i\perp}\nabla\cdot\mathbf{v} - p_{i\perp}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iT\|}\mathbf{b}\right) + \lambda_i\nabla\cdot\boldsymbol{q}_{iT\perp} \simeq 0,$$

$$\frac{1}{2}\mathbf{v}\cdot\nabla p_{i\|} + \frac{1}{2}p_{i\|}\nabla\cdot\mathbf{v} + p_{i\|}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iB\|}\mathbf{b}\right) + \lambda_i\nabla\cdot\boldsymbol{q}_{iB\perp} - 2\lambda_i\boldsymbol{q}_{iB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,$$

- Perpendicular (diamagnetic) heat flux: $\boldsymbol{q}_{s\perp} \equiv \boldsymbol{q}_{sB\perp} + \boldsymbol{q}_{sT\perp} \quad (s = i, e)$

$$\mathbf{q}_{sB\perp} \equiv \frac{m_s}{2}\int\left(v_\| - \bar{v}_\|\right)^2\left(\mathbf{v}_\perp - \bar{\mathbf{v}}_\perp\right) f d^3\mathbf{v} \simeq \frac{1}{e_s B}\mathbf{b}\times\left[\frac{1}{2}p_{s\perp}\nabla\left(\frac{p_{s\|}}{n}\right) + \frac{p_{s\|}\left(p_{s\|} - p_{s\perp}\right)}{n}(\mathbf{b}\cdot\nabla\mathbf{b})\right],$$

$$\mathbf{q}_{sT\perp} \equiv \frac{m_s}{2}\int\left(v_\perp - \bar{v}_\perp\right)^2\left(\mathbf{v}_\perp - \bar{\mathbf{v}}_\perp\right) f d^3\mathbf{v} \simeq \frac{1}{e_s B}\mathbf{b}\times\left[2p_{s\perp}\nabla\left(\frac{p_{s\perp}}{n}\right)\right]$$

- Parallel heat flux: $\boldsymbol{q}_{s\|} \equiv \boldsymbol{q}_{sB\|} + \boldsymbol{q}_{sT\|} \quad (s = i, e)$

$$\mathbf{q}_{sB\|} \equiv \frac{m_s}{2}\int\left(v_\| - \bar{v}_{s\|}\right)^2\left(\mathbf{v}_\| - \bar{\mathbf{v}}_{s\|}\right) f d^3\mathbf{v}, \quad \mathbf{q}_{sT\|} \equiv \frac{m_i}{2}\int\left(v_\perp - \bar{v}_{s\perp}\right)^2\left(\mathbf{v}_\| - \bar{\mathbf{v}}_{s\|}\right) f d^3\mathbf{v}$$

- Parallel heat flux equations for ions $\boldsymbol{q}_{i\parallel} \equiv \boldsymbol{q}_{iB\parallel} + \boldsymbol{q}_{iT\parallel}$

$$\nabla \cdot \left[\left(\mathbf{v} + \frac{\lambda_i}{neB} \nabla p_{i\perp} \times \mathbf{b} \right) q_{iT\parallel} \right] + q_{iT\parallel} \nabla \cdot \left(\mathbf{v} + \frac{\lambda_i}{neB} \nabla p_{i\perp} \times \mathbf{b} \right)$$

$$+ \frac{p_{i\parallel}}{m_i} \mathbf{b} \cdot \nabla \left(\frac{p_{i\perp}}{n} \right) - \frac{p_{i\perp} \left(p_{i\parallel} - p_{i\perp} \right)}{m_i n B} \mathbf{b} \cdot \nabla B \simeq 0,$$

$$\nabla \cdot \left[\left(\mathbf{v} + \frac{\lambda_i}{neB} \nabla p_{i\perp} \times \mathbf{b} \right) q_{iB\parallel} \right] + \frac{3 p_{i\parallel}}{2 m_i} \mathbf{b} \cdot \nabla \left(\frac{p_{i\parallel}}{n} \right) \simeq 0,$$

- Parallel heat flux equations for mass-less electrons, $m_e \approx 0$

$$\mathbf{B} \cdot \nabla \left(p_{e\parallel} / n \right) = 0, \quad \mathbf{B} \cdot \nabla \left[\left(p_{e\parallel} / p_{e\perp} - 1 \right) B \right] = 0$$

-Fluid closure condition:

$$\int d^3\mathbf{v} \left(v_i - \overline{v}_i \right) \left(v_j - \overline{v}_j \right) \left(v_k - \overline{v}_k \right) \left(v_l - \overline{v}_l \right) f\left(\mathbf{x}, \mathbf{v}, t \right)$$
$$= \frac{1}{n} \left[\int d^3\mathbf{v} \left(v_{[i} - \overline{v}_{[i} \right) \left(v_j - \overline{v}_j \right) f\left(\mathbf{x}, \mathbf{v}, t \right) \right] \left[\int d^3\mathbf{v} \left(v_k - \overline{v}_k \right) \left(v_{l]} - \overline{v}_{l]} \right) f\left(\mathbf{x}, \mathbf{v}, t \right) \right]$$

- ➢ Kinetic effects in the fourth-order moments are neglected

- Ion gyroviscous force for collisionless magnetized plasmas

$$\nabla\cdot\Pi_i^{gv} = \nabla\cdot\left(\sum_{N=1}^{5}\Pi_i^{gvN}\right),$$

$$\nabla\cdot\Pi_i^{gv1} \simeq -m_i n\mathbf{v}_{*i}\cdot\nabla\mathbf{v} - \nabla\chi_v - \nabla\times\left[\frac{m_i p_{i\perp}}{2eB^2}(\nabla\cdot\mathbf{v})\mathbf{B}\right],$$

$$\nabla\cdot\Pi_i^{gv2} \simeq -\nabla\chi_q - \nabla\times\left[\frac{m_i}{4eB^2}(\nabla\cdot\mathbf{q}_{iT\perp})\mathbf{B}\right],$$

$$\nabla\cdot\Pi_i^{gv3} \simeq \nabla\times\left\{\mathbf{B}\times\left[\frac{m_i}{eB^2}(\mathbf{c}+\mathbf{d})\right]\right\}, \qquad \nabla\cdot\Pi_i^{gv4} \simeq \nabla\cdot\Pi_i^{gv5} \simeq 0,$$

$$\mathbf{v}_{*i} \equiv -\frac{1}{en}\nabla\times\left(\frac{p_{i\perp}}{B^2}\mathbf{B}\right), \qquad \chi_v \equiv \frac{m_i p_{i\perp}}{2eB^2}\mathbf{B}\cdot(\nabla\times\mathbf{v}), \qquad \chi_q \equiv \frac{m_i}{5eB^2}\mathbf{B}\cdot(\nabla\times\mathbf{q}_{i\perp}),$$

$$\mathbf{c} \equiv -\left(\frac{p_{i\|} - p_{i\perp}}{B}\right)\nabla\times\left(\frac{1}{en}\nabla p_{i\perp}\right), \qquad \mathbf{d} \equiv \nabla\times\left[\left(2\mathbf{q}_{iB\perp} - \frac{1}{2}\mathbf{q}_{iT\perp}\right)\times\mathbf{b}\right]$$

Orderings for reduced MHD

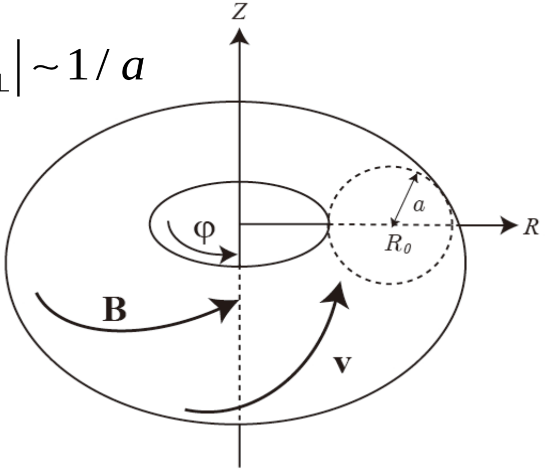
- Compressible high-β tokamak and slow dynamics orderings
  - Large aspect ratio and high-β tokamak

$$\varepsilon \equiv a/R_0 \ll 1, \quad B_p \sim \varepsilon B_0, \quad p \sim \varepsilon\left(B_0^2/\mu_0\right), \quad \left|\nabla_{\parallel}\right| \sim 1/R_0, \quad \left|\nabla_{\perp}\right| \sim 1/a$$

a, R_0 : minor and major radii of a torus

B_p, B_0 : poloidal and toroidal magnetic fields

p : pressure



  - Weak compressibility $\nabla \cdot \mathbf{v}_{MHD} \sim \varepsilon v_{MHD}/a, \quad \left(\boldsymbol{v} \equiv \boldsymbol{v}_{MHD} + \boldsymbol{v}_{di}\right)$

    The fast magnetosonic wave is eliminated.
  - Flow velocity for slow dynamics $\delta \equiv \rho_i / a \ll 1, \quad \rho_i$: ion Larmor radius

$$v \sim v_{MHD} \sim v_{di} \sim \delta v_{th}, \quad \left|\nabla \cdot \Pi^{gv}\right| \sim \delta^2 \left|\nabla p\right| \sim \delta^2 m_i n v_{th}^2 \quad q \sim pv \sim \delta p v_{th}$$

- Strong pressure anisotropy: $\left|p_{\parallel} - p_{\perp}\right| \sim p$
- Parallel heat flux can not be neglected: $\left|q_{\parallel} \sim q_{\perp}\right| \sim pv$

  - Poloidal sound velocity $v_{\varphi} \sim v_p \sim \left(B_p/B_0\right)\sqrt{\gamma p/\rho} \sim \varepsilon\sqrt{\gamma p/\rho} \sim \varepsilon v_{th}$:

$$\rho v^2 \sim \left(B_p/B_0\right)^2 \gamma p \sim \underline{\varepsilon^3}\left(B_0^2/\mu_0\right) \quad \Rightarrow \quad \delta \sim \varepsilon$$

    - Transition between sub- and super-poloidal-sonic flow appears.
    - Higher-order terms should be taken into account.

Equilibrium flow comparable to poloidal sound velocity

[Ito and Nakajima, NF **51**, 123006 (2011)]

- Reduced equilibrium equations

- Axisymmetric equilibria:

$$\partial/\partial\varphi = 0, \quad \mathbf{B} = \nabla\psi \times \nabla\varphi + I\nabla\varphi \qquad \left[\text{cylindrical geometry}\left(R,\varphi,Z\right)\right]$$

- Asymptotic expansions: $f = f_0 + f_1 + f_2 + f_3 + \cdots,\ f_1 \sim \varepsilon f_0,\ f_2 \sim \varepsilon^2 f_0,\ f_3 \sim \varepsilon^3 f_0,$

- Lowest order quantities are functions of ψ_1

$$\Phi_1 = \Phi_1(\psi_1), \quad n_0 = n_0(\psi_1), \quad p_{i\|1} = p_{i1}(\psi_1), \quad p_{i\perp 1} = p_{i\perp 1}(\psi_1),$$
$$p_{e\|1} = p_{e\|1}(\psi_1), \quad p_{e\perp 1} = p_{e\perp 1}(\psi_1), \quad I_1 = I_1(\psi_1)$$

- Higher-order quantities are determined by ψ_1 and ψ_2

$$p_{s\{\|,\perp\}2} = p'_{s\{\|,\perp\}1}\psi_2 + \left(x/R_0\right)C_{s\{\|,\perp\}}\left(\psi_1\right) + P_{s\{\|,\perp\}2*}\left(\psi_1\right),$$

Shift from magnetic surfaces

$$\Phi_2 = \Phi_1'\psi_2 + \left(x/R_0\right)C_\Phi\left(\psi_1\right) + \Phi_{2*}\left(\psi_1\right), \qquad n_1 = n_0'\psi_2 + \left(x/R_0\right)C_n\left(\psi_1\right) + n_{1*}\left(\psi_1\right),$$

$$v_\| \equiv \left(x/R_0\right)C_{v\|}\left(\psi_1\right) + v_{\|*}\left(\psi_1\right),$$

$$q_{s\|} \equiv \left(x/R_0\right)C_{sq\|}\left(\psi_1\right) + q_{s\|*}\left(\psi_1\right), \qquad q_{sB\|} \equiv \left(x/R_0\right)C_{sqB\|}\left(\psi_1\right) + q_{sB\|*}\left(\psi_1\right),$$

- $P_{s\{\|,\perp\}*}, v_{\|*}, \Phi_{2*}, n_{1*}, q_{s\|*}, q_{sB\|*}$ are arbitrary functions of ψ_1
- $C_{s\{\|,\perp\}}, C_{v\|}, C_{\Phi}, C_n, C_{sq\|}, C_{sqB\|}, C_I$ are obtained from the equations for $p_{s\{\|,\perp\}2}, v_{\|}, \Phi_2, n_1, q_{s\{\|,\perp\}}, q_{i\mathbf{B}\{\|,\perp\}}, I_2$

- Poloidal force balance

$$p_{i\perp 1} + p_{e\perp 1} + \frac{B_0}{\mu_0 R_0} I_1 = \text{const.}, \quad p_{i\perp 2} + p_{e\perp 2} + \frac{B_0}{\mu_0 R_0} I_2 - \left(\frac{x}{R_0}\right) \sum_{s=i,e} (p_{s\|1} - p_{s\perp 1}) \equiv g_*(\psi_1),$$

- Radial force balance yields the Grad-Shafranov (GS) type equations

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 = -\mu_0 R_0^2 \left[\left(\frac{x}{R_0}\right)\sum_{s=i,e}(p'_{s\|1} + p'_{s\perp 1}) + g'_*\right] - \left(\frac{I_1^2}{2}\right)', \qquad (x \equiv R - R_0, \quad x \sim a)$$

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_2 + \left[\mu_0 R_0^2 \left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(p''_{s\perp 1} + p''_{s\|1}\right) + \mu_0 R_0^2 g''_* + \left(\frac{I_1^2}{2}\right)''\right]\psi_2 = \frac{1}{R}\frac{\partial \psi_1}{\partial R} + F\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 + F'\frac{|\nabla \psi_1|^2}{2}$$

$$-\mu_0 R_0^2 \left[E'_* + \left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(P'_{s\perp 2*} + P'_{s\|2*}\right) + \frac{1}{2}\left(\frac{x}{R_0}\right)^2 \sum_{s=i,e}\left(p'_{s\perp 1} + p'_{s\|1} + C'_{s\perp 1} + C'_{s\|1}\right)\right].$$

- GS equation for ψ_2 includes the effect of flow, FLR and pressure anisotropy:

$$F(\psi_1) \equiv \left[V_E - \underline{(\lambda_H - \lambda_i)} V_{di}\right]\left(V_E - \lambda_H V_{di}\right) + \underline{\sum_{s=i,e} \frac{p_{s\|1} - p_{s\perp 1}}{B_0^2/\mu_0}}$$

Gyroviscous cancellation

Pressure anisotropy

$V_E(\psi_1), V_{di}(\psi_1)$: Poloidal Alfven Mach numbers of the $E\times B$ drift and the ion diamagnetic drift velocities

Analytic solution for single-fluid MHD

[A. Ito and N. Nakajima, Plasma Phys. Control. Fusion **51** 035007 (2009), **61** 029501 (2019)]

- Reduced GS equations for MHD equilibria can be solved analytically for linear profiles

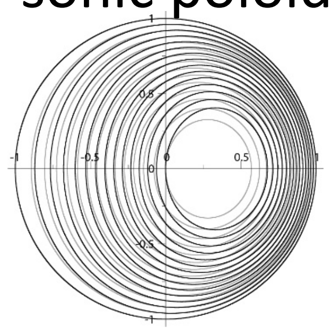
$$p_1 = \varepsilon \left(B_0^2 \big/ \mu_0 \right) p_{1c} \overline{\psi}_1, \quad g_* + I_1^2 \big/ 2\mu_0 R_0^2 = \varepsilon^2 \left(B_0^2 \big/ \mu_0 \right) g_c \overline{\psi}_1,$$

$$M_{Ap}^2 \equiv \mu_0 m_i n_0 \left(R_0 \Phi_1{}' \big/ B_0 \right)^2 = \varepsilon M_{Apc}^2 \overline{\psi}_1, \quad \overline{\psi}_1 \equiv \psi_1 \big/ \psi_c, \quad \varepsilon \overline{\psi}_2 \equiv \psi_2 \big/ \psi_c$$

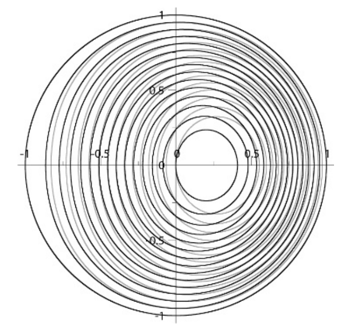
$E_* = p_{2*} = p_{3*} = 0,$ $\quad \psi_c, p_{1c}, g_c$ and M_{Apc} are constant.

(M_{Ap} : poloidal Alfven Mach number of poloidal flow)

-Sub-sonic poloidal flow



- Super-sonic poloidal flow



Black: pressure isosurfaces

Gray: Magnetic flux surfaces

- Magnetic surfaces are modified due to flow
- The pressure maximum is shifted outwards for sub-sonic flow and inwards for super-sonic flow

Analytic solution for single-fluid equilibrium with flow and pressure anisotropy: $(\lambda_H, \lambda_i) = (0, 0)$

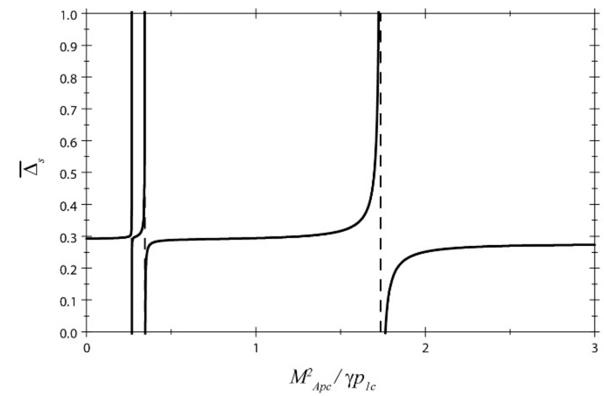
[A. Ito and N. Nakajima, J. Phys. Soc. Jpn **82** 064502 (2013), **88** 028001 (2019)]

- Anisotropic ion pressure in the presence of the parallel heat flux, $\lambda_{i\|} = 1$
  - Singularity

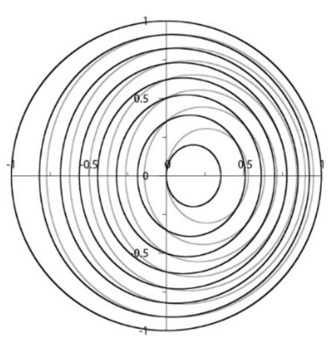
$$M_{Apc}^2 = \frac{1}{2}\left(6p_{i\|1c} + p_{e\|1c} \pm \sqrt{24p_{i\|1c}^2 + p_{e\|1c}^2}\right)$$

(slow magnetosonic and ion acoustic waves)

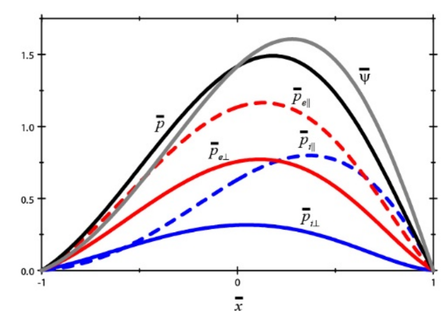
$$M_{Apc}^2 = p_{i\|1c}$$ (ion acoustic wave)



- Pressure contour (black) and magnetic flux surfaces (gray)



- Pressure profiles



Numerical solution for the FLR two-fluid model

- Profiles of free functions:

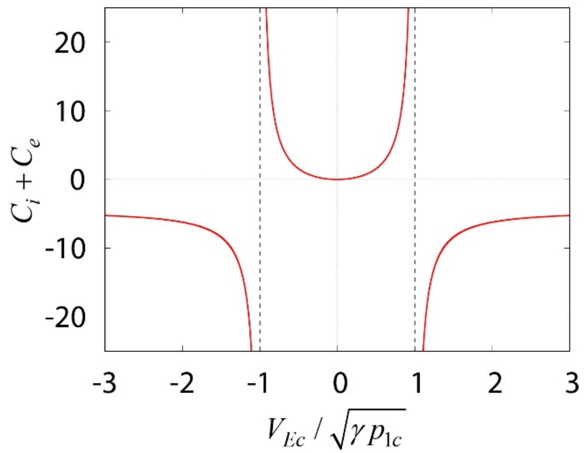
$$p_{s1\|} = \varepsilon\left(B_0^2/\mu_0\right) p_{s1\|c} \left(\psi_1/\psi_c\right)^4, \quad p_{s1\perp} = \varepsilon\left(B_0^2/\mu_0\right) p_{s1\perp c} \left(\psi_1/\psi_c\right)^4, \quad n_0 = n_{0c}\left(\psi_1/\psi_c\right)^2$$

$$g_* + \frac{I_1^2}{2\mu_0 R_0^2} = \varepsilon^2\left(B_0^2/\mu_0\right) g_c \left(\psi_1/\psi_c\right), \quad V_E = \sqrt{\varepsilon} V_{Ec} \left(\psi_1/\psi_c\right)^2, \quad V_{di} = 4\sqrt{\varepsilon} V_{dic}\, p_{i\perp c} \left(\psi_1/\psi_c\right)^2,$$

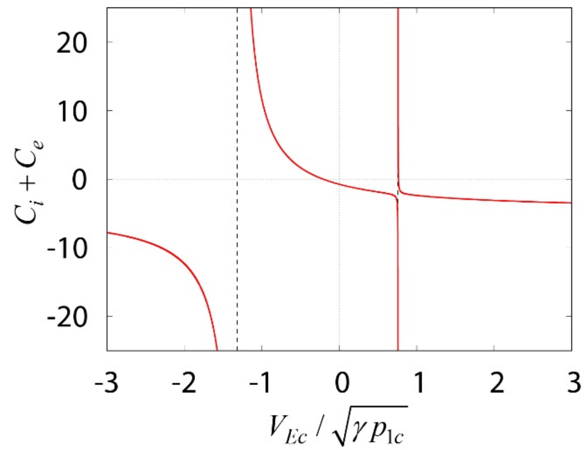
$$\Rightarrow \quad C_{s\{\|,\perp\}} = \varepsilon\left(B_0^2/\mu_0\right) C_{s\{\|,\perp\}c} \left(\psi_1/\psi_c\right)^4, \quad C_{s\{\|,\perp\}c} = \text{const.} \qquad p_{1c} = \sum_{s=i,e} \frac{p_{s1\|c} + p_{s1\perp c}}{2}$$

- Dependence of higher-order terms on V_{Ec} for $\lambda_{i\|} = 0$ $\left(V_{dic} = \sqrt{0.01\gamma p_{1c}}\right)$
  - Isotropic pressure
  - Adiabadic electron pressure

MHD



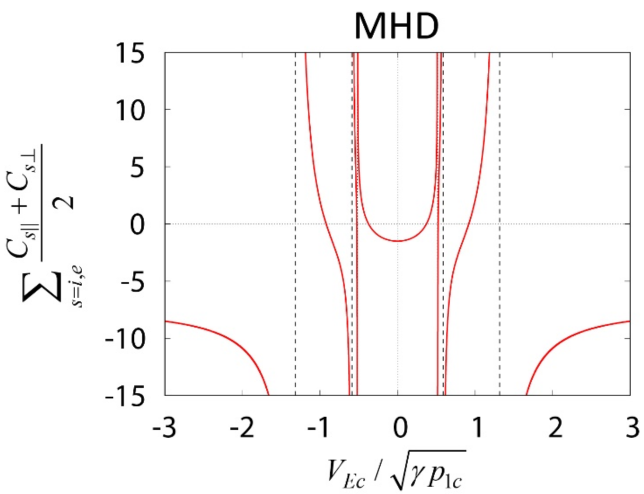
FLR+TF

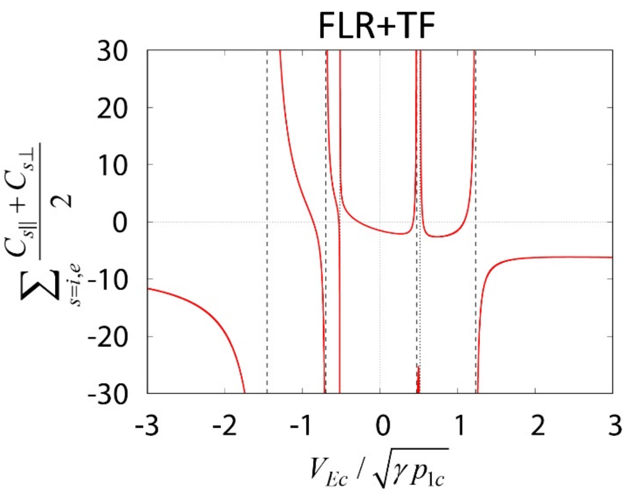


- 2 singular points (1 sound wave)
- Symmetric

- 2 singular points (1 sound wave)
- Asymmetric due to ion diamag. flow

- Dependence of higher-order terms on V_{Ec} for $\lambda_{i\|}=1$ $\left(V_{dc}=-\sqrt{0.01\gamma p_{1c}}\right)$





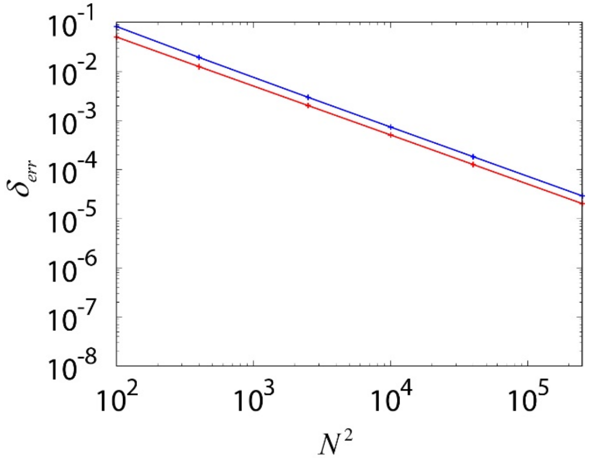
- 6 singular points (3 sound waves)
- Symmetric

- 6 singular points (3 sound waves)
- Asymmetric due to ion diamag. flow

- Singular points appear due to the ordering $\left|p-m_i n v_\perp^2\right|/p\sim 1.$
- Small scale effects on regular solutions for single-fluid MHD are studied.

- Boundary conditions:
  - Circular cross-section, $\psi_1(1,\theta)=0, \quad \psi_2(1,\theta)=0.$
  - Up-down symmetry
- Finite element method
  - GS Eq. for ψ_1: nonlinear, solved iteratively
  - GS Eq. for ψ_2: linear, solved by substituting ψ_1
- Numerical solutions
  - Finite element method with N^2 meshes
  - Linear profile: benchmarked with analytic solution

$$\delta_{err1}^2 = \frac{\sum_{i=1}^{N(N+1)}[\overline{\psi}_{1i} - \overline{\psi}_1(\overline{r}_i,\theta_i)]^2}{\sum_{i=1}^{N(N+1)}\overline{\psi}_{1i}^2}, \qquad \delta_{err2}^2 = \frac{\sum_{i=1}^{N(N+1)}[\overline{\psi}_{2i} - \overline{\psi}_2(\overline{r}_i,\theta_i)]^2}{\sum_{i=1}^{N(N+1)}\overline{\psi}_{1i}^2},$$



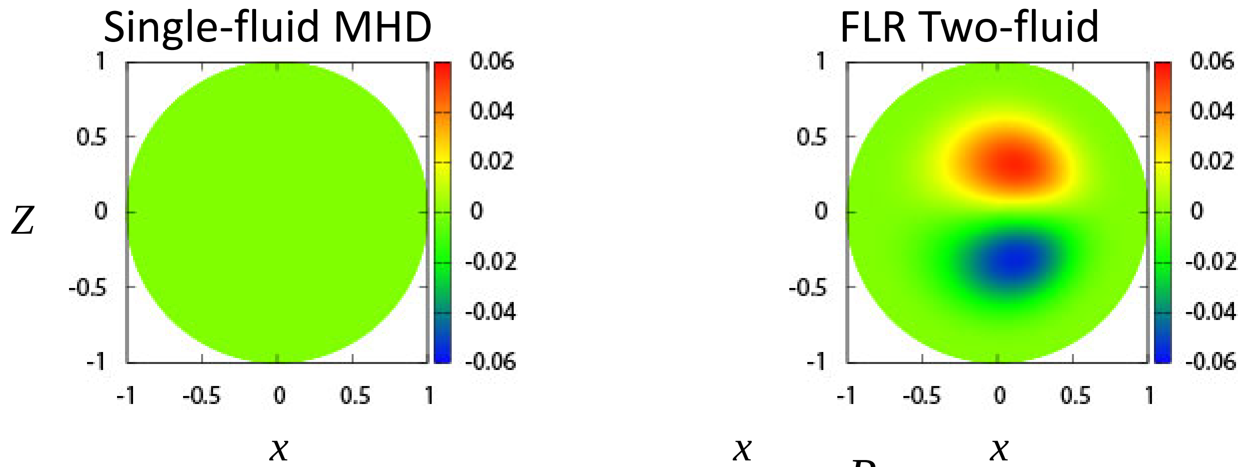
$\overline{\psi}_{1i}, \overline{\psi}_{2i}$: Numerical solutions at each grid points

$\overline{\psi}_1(\overline{r}_i,\theta_i), \overline{\psi}_2(\overline{r}_i,\theta_i)$: Analytic solutions at each grid points

The analytic solution enables the benchmark of numerical solution.

- Shift of isosurfaces of ion stream function from magnetic surfaces:

$$\left(\nabla\psi\times\nabla\Psi\right)\cdot\left(R\nabla\varphi\right)$$

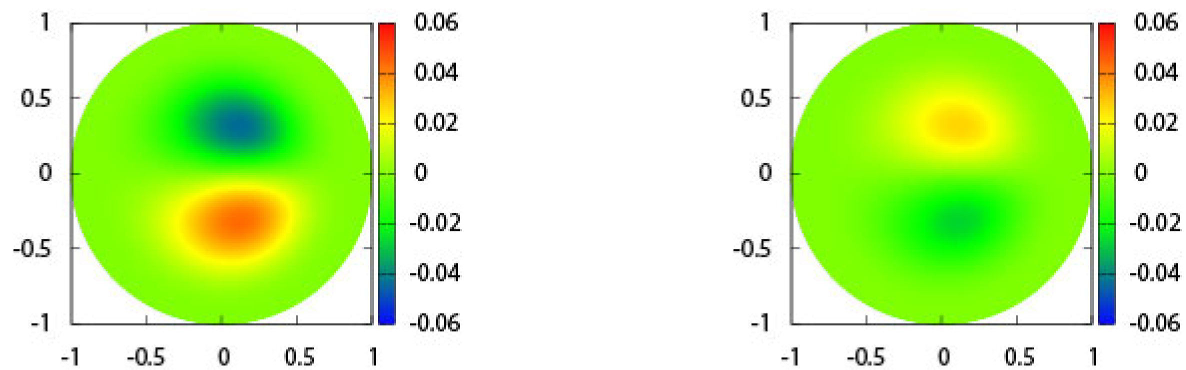


$$\Psi_2=\Psi_1'\psi_2+\left(x/R_0\right)\lambda_H C_\Psi\left(\psi_1\right)+\Psi_{2*}\left(\psi_1\right),\quad C_\Psi\left(\psi_1\right)=-\frac{R_0}{eB_0}\left[C_{i\perp}-\left(p_{i\parallel}-p_{i\perp}\right)+C_{e\parallel}+p_{e\parallel}-p_{e\perp}\right]$$

Isosurfaces of ion stream function shift from magnetic surfaces due to two-fluid effect, but it also depends on FLR effects.

- Shift of pressure isosurfaces occurs due to flow or pressure anisotropy even in the single-fluid model

$$\left(\nabla\psi\times\nabla p_{i\parallel}\right)\cdot\left(R\nabla\varphi\right)$$



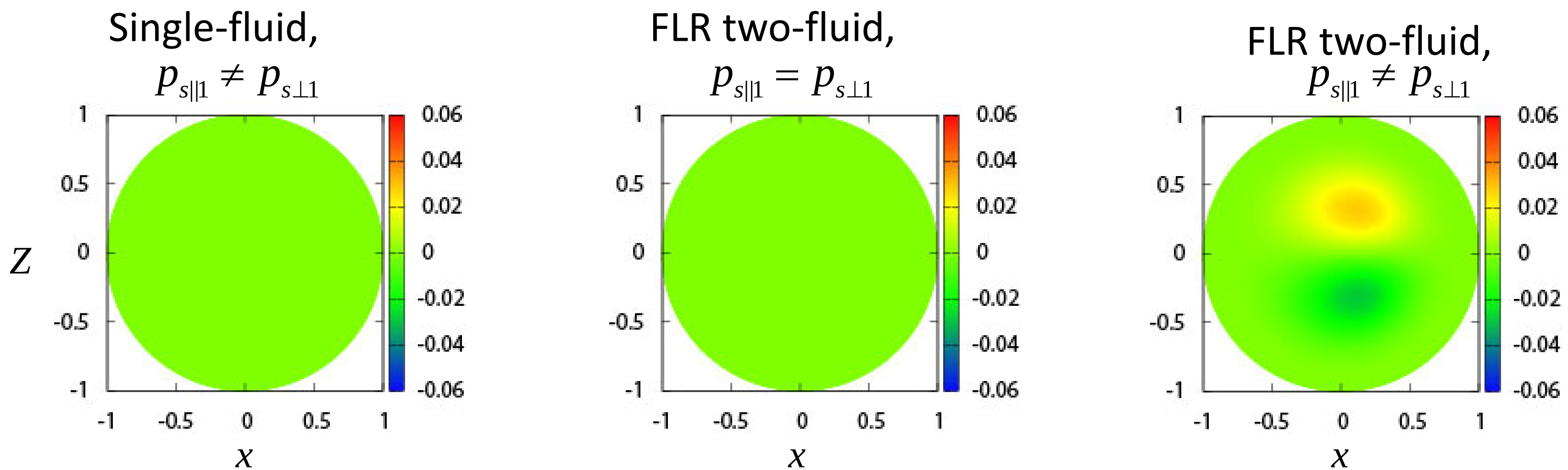
- Effect of pressure anisotropy

  - Electron stream function

$$\Psi_{e2} = \Psi'_{e1}\psi_2 + (x/R_0)\lambda_H C_{\Psi e}(\psi_1) + \Psi_{e2*}(\psi_1), \quad C_{\Psi e}(\psi_1) = -\frac{R_0}{eB_0}\underline{\left[C_{e\parallel} - C_{e\perp} + 2\left(p_{e\parallel 1} - p_{e\perp 1}\right)\right]}$$

$$(\nabla\psi \times \nabla\Psi_e)\cdot(R\nabla\varphi)$$

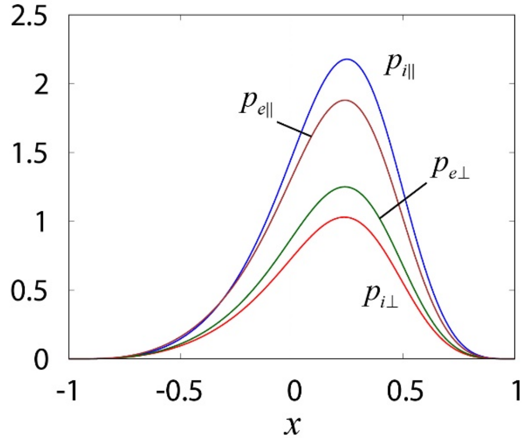
Pressure anisotropy



Shift of the isosurfaces of the electron stream function occur in the presence of both two-fluid effects and pressure anisotropy

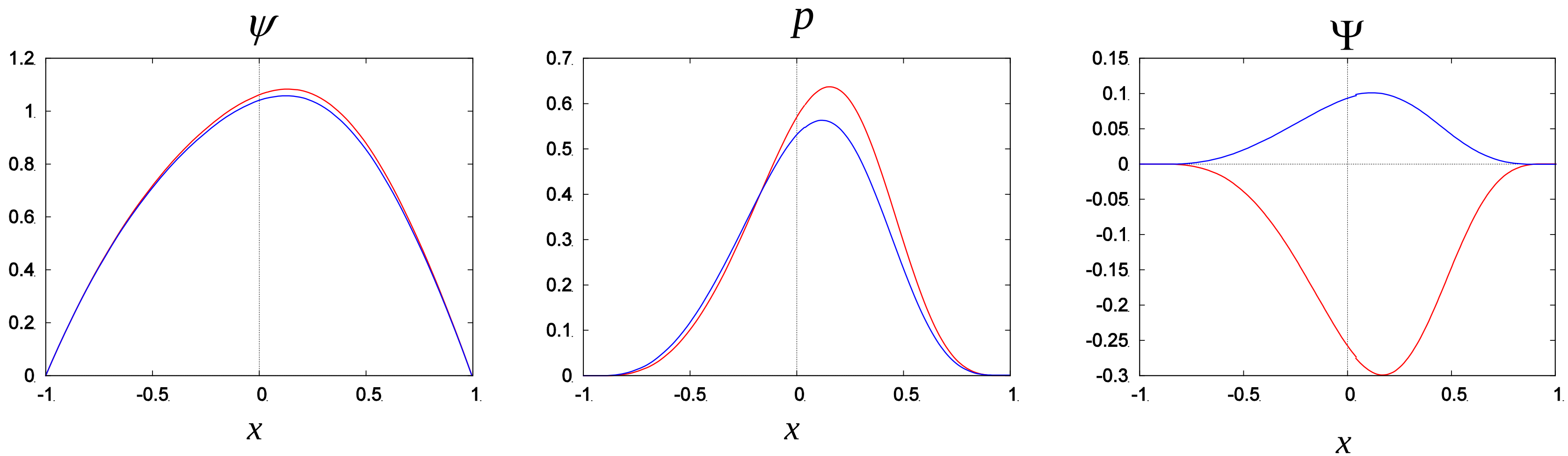
➢ Pressure profiles in the midplane

$$\left(p_{i\|1c} = 2.0\, p_{i\perp 1c},\ p_{e\|1c} = 1.5\, p_{e\perp 1c},\ p_{i\|1c} + p_{i\perp 1c} = p_{e\|1c} + p_{e\perp 1c} \right)$$



Anisotropic pressures for ions and electrons are self-consistently obtained.

- Solutions depend on the sign of $E \times B$ flow compared to that of ion diamagnetic flow



Red: same direction

Blue: opposite direction

Reduced MHD equations for stability in the presence of poloidal flow

- Reduced-MHD equations with $\partial/\partial t \neq 0, \partial/\partial\varphi \neq 0$

  - must include equilibria with poloidal-sonic flow when $\partial/\partial t = 0, \partial/\partial\varphi = 0$

  - require the energy conservation up to $O\left[\varepsilon^3\left(B_0^2/\mu_0\right)\right]$

  - are needed for stability of toroidal equilibria with strong poloidal flow

- Reduced MHD equations with higher order terms

  - We modify the reduced equations found by Strauss [NF **23**, 649 (1983)] to apply for high-beta plasmas with dynamics of slow magnetosonic wave and non-constant density.

- Reduced MHD equations $\mathbf{v} \equiv \left(1 + x/R_0\right)\nabla U \times \left(\mathbf{B}/B\right) + v_\parallel\left(\mathbf{B}/B\right), \quad \mathbf{B} \equiv \mathbf{H} \times \nabla\varphi + I\nabla\varphi,$

$$\mathbf{H} \equiv \nabla\psi - \frac{1}{\xi}\frac{\partial^2 F}{\partial\varphi\partial\Theta}\nabla\xi\left(R,Z\right) + \xi\frac{\partial^2 F}{\partial\varphi\partial\xi}\nabla\Theta\left(R,Z\right).$$

Leading order force balance: $p_1 + \frac{B_0}{\mu_0 R_0} I_1 = const.$

Leading order pressure equation: $\mathbf{v}\cdot\nabla p_1 = 0$

$$\frac{\partial\rho}{\partial t}+\frac{R}{R_0 B}\left[\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi+I\nabla U\times\nabla\varphi\right]\cdot\nabla\rho$$
$$+\frac{v_{\parallel}R}{R_0 B_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla\rho$$
$$+\rho\nabla\left(\frac{R}{R_0 B}\right)\cdot\left[\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi+I\nabla U\times\nabla\varphi\right]$$
$$+\frac{\rho R}{R_0 B}\left[\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left(\nabla U\cdot\nabla\varphi\right)-\nabla\varphi\cdot\nabla\left(\mathbf{H}\cdot\nabla U\right)+\nabla I\cdot\left(\nabla U\times\nabla\varphi\right)\right]$$
$$=0$$

$$\left[\frac{\partial}{\partial t}+R_0\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\right]\left[\nabla_\perp\cdot\left(\rho R_0\nabla_\perp U\right)\right]+\left\{\rho,\frac{R_0^2\left|\nabla_\perp U\right|^2}{2}\right\}-\left\{R^2-R_0^2,p\right\}$$
$$+\left[\mathbf{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left(Rj_\varphi\right)+\mu_0\frac{j_\varphi}{B_0}\frac{\partial p}{\partial\varphi}+\frac{1}{B_0R_0}\frac{\partial\mathbf{H}}{\partial\varphi}\cdot\nabla p=0,$$

$$\frac{\partial\psi}{\partial t}=\frac{R^2}{R_0}\nabla\varphi\cdot\left(\nabla U\times\mathbf{H}\right)+B_0\frac{\partial U}{\partial\varphi}-\frac{\partial\widehat{\Phi}}{\partial\varphi},$$

$$\rho\left[\frac{\partial}{\partial t}+R_0\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\right]v_\parallel+\frac{R}{B_0R_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla p=0,$$

$$\frac{\partial p}{\partial t}+\frac{v_\parallel R}{B_0R_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla p+\left(R/BR_0\right)\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]\cdot\nabla p$$
$$+\gamma p\left[\mathbf{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left[\frac{v_\parallel R}{B_0R_0}\left(1+\frac{p}{B_0^2/\mu_0}\right)\right]+\gamma p\nabla\left(R/BR_0\right)\cdot\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]$$
$$+\frac{\gamma pR}{BR_0}\left[\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left(\nabla U\cdot\nabla\varphi\right)-\nabla\varphi\cdot\nabla\left(\mathbf{H}\cdot\nabla U\right)+\nabla I\cdot\left(\nabla U\times\nabla\varphi\right)\right]=0,$$

$$\frac{1}{R^2}\frac{\partial I}{\partial t}+\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\left(\frac{I^2}{R_0RB}\right)+\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left[\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\right]$$
$$+\left(\mathbf{H}\times\nabla\varphi\right)\cdot\nabla\left(\frac{\mathbf{H}\cdot\nabla U}{R_0RB}\right)+\frac{\mu_0}{R_0RB}\left(\mathbf{H}\cdot\nabla U\right)\left(\nabla\varphi\cdot\nabla p\right)=0,$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial F}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2 F}{\partial\Theta^2}=\frac{\mu_0 p}{\xi J B_0},\quad J\equiv R_0\left(\nabla\xi\times\nabla\Theta\right)\cdot\nabla\varphi$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\widehat{\Phi}}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2\widehat{\Phi}}{\partial\Theta^2}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\mu_0 p}{B_0}\frac{\partial U}{\partial\xi}\right)+\frac{1}{\xi}\frac{\partial}{\partial\Theta}\left(\frac{\mu_0 p}{B_0}\frac{\partial U}{\partial\Theta}\right),$$

- By taking asymptotic expansions for axisymmetric equilibria, reduced equilibrium equations are reproduced
- Shear Alfven and slow magnetosonic waves are found in the homogeneous, cylindrical limit

- Energy conservation

$$\frac{1}{2}\frac{\partial}{\partial t}\int d^3\mathbf{x}\left\{\left[\rho\left(\left|\nabla U\right|^2+v_{\parallel}^2\right)+\frac{1}{\mu_0 R^2}\left(\left|\mathbf{H}\right|^2+I^2\right)+\frac{2p}{\gamma-1}\right]\right\}=0,$$

can be shown with asymptotic expansions in ε up to $O\left[\varepsilon^3\left(B_0^2/\mu_0\right)\right]$.

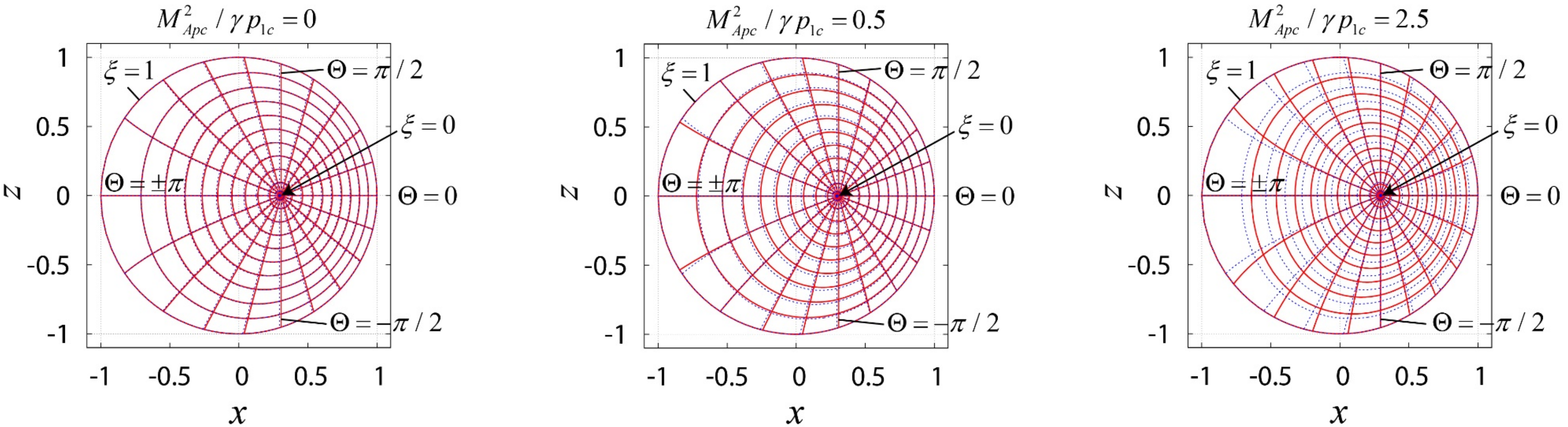
Magnetic flux coordinates

[A. Ito and N. Nakajima, PPCF **61** 105006 (2019)]

- Flux coordinates obtained from analytic MHD equilibrium with flow

Flux coordinates (ξ,Θ) for different poloidal flow velocities

(Red: $\psi_1+\varepsilon\psi_2$, Blue: ψ_1)



- Flux coordinates are modified due to poloidal flow
- Flux coordinates will be used for stability analysis

Summary

- Reduced equations for two-fluid equilibria with flow
  - Two-fluid equilibria with toroidal and poloidal flow, ion FLR, pressure anisotropy and parallel heat flux have been derived from the fluid moment equations for collisionless magnetized plasmas.
- Analytic solution for single-fluid equilibria
  - The solution indicates the modification of the magnetic flux and the departure of the pressure surfaces from the magnetic surfaces due to flow.
  - Complicated characteristics in the region around the poloidal sound velocity due to pressure anisotropy and the parallel heat flux have been found.

- Numerical solution for two-fluid equilibria with ion FLR
  - The isosurfaces of the magnetic flux, the pressure and the ion stream function do not coincide with each other.
  - Pressure anisotropy associated with parallel heat flux has been included in the numerical code.
  - Solutions depend on the direction of $E \times B$ flow compared to that of ion diamagnetic flow.
- Reduced MHD equations
  - We have derived time-dependent reduced MHD equations consistent with the high-beta tokamak equilibrium with strong poloidal flow.
- Flux coordinates in equilibrium with flow
  - We have obtained modified flux coordinates by adding second order magnetic flux in the presence of poloidal flow.